Inequalities for f*-vectors of lattice polytopes

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Geometry meets Combinatorics in Bielefeld, September 2022

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We started this project at the online workshop *Research Encounters in Algebraic and Combinatorial Topics* (REACT 2021).

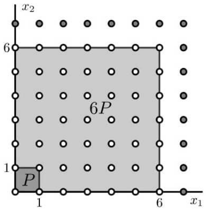
- $P \subset \mathbb{R}^d$ is a d-dimensional lattice polytope.
- $nP := \{np : p \in P\}$ is the n-th dilate of P, $n \in \mathbb{N}$.

Definition

The function

$$\mathrm{ehr}_P(n) := |nP \cap \mathbb{Z}^d|$$

is a polynomial in n (Ehrhart, 1962), known as the **Ehrhart polynomial** of P.



(source: Computing the Continuous Discretely, M. Beck & S. Robins, Springer, 2007)

Change of basis

Study $\mathrm{ehr}_P(n)$ in different basis:

$$\left\{\binom{n+d}{d}, \binom{n+d-1}{d}, \dots, \binom{n}{d}\right\} \text{ and } \left\{\binom{n-1}{0}, \binom{n-1}{1}, \dots, \binom{n-1}{d}\right\}$$

- $\mathrm{ehr}_P(n) = \sum_{k=0}^{d} h_k^* \binom{n+d-k}{d}$ and $\mathrm{ehr}_P(n) = \sum_{k=0}^{d} f_k^* \binom{n-1}{k}$.

Proposition

For every lattice polytope P of dimension d, for every $0 \leq k \leq d$,

$$h_k^* \geq 0, \qquad f_k^* \geq 0.$$

Stanley's nonnegativity theorem — *Breuer (2012)*

Let $P = [0,1]^2$ be the 2-dimensional unit cube. Then

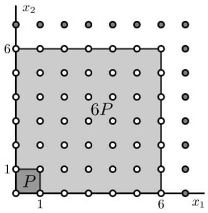
$$\mathrm{ehr}_P(n) = \left|nP \cap \mathbb{Z}^2\right| = \left|[0,n]^2 \cap \mathbb{Z}^d\right| = (n+1)^2,$$

and
$$\begin{cases} (n+1)^2 = h_0^*\binom{n+2}{2} + h_1^*\binom{n+1}{2} + h_2^*\binom{n}{2} \\ (n+1)^2 = f_0^*\binom{n-1}{0} + f_1^*\binom{n-1}{1} + f_2^*\binom{n-1}{2} \end{cases},$$

hence

$$(h_0^*, h_1^*, h_2^*) = (1,1,0),$$

$$(f_0^*, f_1^*, f_2^*) = (4,5,2).$$



The 6^{th} dilate of $P = [0,1]^2$.

Generating function

The f^*- and h^*-vector can also be defined through the **Ehrhart series** of P:

$$\mathrm{Ehr}_P(z) := 1 + \sum_{n \geq 1} \mathrm{ehr}_P(n) z^n = \frac{\sum_{k=0}^{d} h_k^* z^k}{(1-z)^{d+1}}$$

$$= \sum_{k=-1}^{d} f_k^* \left(\frac{z}{1-z} \right)^{k+1},$$

where we let $f_{-1}^* := 1$.

Let $P := \Delta$ be a unimodular d-dimensional simplex, i.e., lattice equivalent to $\operatorname{conv}\{(0,\dots,0),(1,\dots,0),\dots,(0,\dots,1)\} \subset \mathbb{R}^d$.

Example

Then, for Δ we have $[h_0^*,\dots,h_d^*] = [1,0,\dots,0]$ and

$$[f_{-1}^*, f_0^*, \dots, f_d^*] = \left[1, \binom{d+1}{1}, \binom{d+1}{2}, \dots, \binom{d+1}{d+1}\right].$$

Observation: Notice that $[f_{-1}^*, f_0^*, \dots, f_d^*]$ is **symmetric**.
This is the only lattice polytope with symmetric f^*-vector!

- Does $[f_{-1}^*, f_0^*, \dots, f_d^*]$ look familiar?

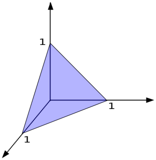


image source: Wikipedia

Definition

For a d-dimensional polytopal complex C, let

$$f(C) := \{f_{-1}, f_0, \dots, f_d\}, \quad f_k = \#\{k\text{-dimensional faces in } C\}.$$

For a d-dimensional polytope P, let

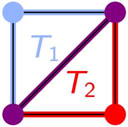
$$f(P) := \{f_{-1}, f_0, \dots, f_{d-1}\}, \quad f_k = \#\{k\text{-dimensional faces in } P\}.$$

Connection: If P admits a unimodular triangulation T then

$$f^*(P) = f(T).$$

Example:

$P = [0,1]^2$, $T = T_1 \cup T_2$



$$f^*(P) = (f^*_{-1}, f^*_0, f^*_1, f^*_2) = (1, 4, 5, 2)$$

$$f(T) = (f_{-1}, f_0, f_1, f_2) = (1, 4, 5, 2)$$

There are several inequalities holding among the coefficients of the h^*-vector of a d-dimensional lattice polytope P, for example:

- $h_0^* + h_1^* + \cdots + h_{k+1}^* \geq h_d^* + h_{d-1}^* + \cdots + h_{d-k}^*$
  for $k = 0, \ldots, \lfloor \frac{d}{2} \rfloor - 1$ (Hibi, 1990).
- If $h_d^* \neq 0$ then

$$h_0^* + h_1^* + \cdots + h_k^* \leq h_d^* + h_{d-1}^* + \cdots + h_{d-k}^*$$

  for $k = 0, ..., d$ (Stanley, 1991),

  and $1 \leq h_1^* \leq h_k^*$ for $k = 2, ..., d-1$ (Hibi, 1994).

▶ Is $h^*(P)$ **unimodal** for every lattice polytope P?
(i.e., $\exists\, j : h_0^* \leq \cdots \leq h_{p-1}^* \leq h_p^* \geq h_{p+1}^* \geq \cdots \geq h_d^*$) No!

Higashitani (2012) constructed an infinite family of simplices with nonunimodal h^*-vectors.

Example

The simplex of dimension 15 such that

$$\Delta_w = \mathrm{conv}\{0, e_1, e_2, \ldots, e_{14}, w\},$$

$$\text{where} \quad w = (\underbrace{1, 1, \ldots, 1}_{7}, \underbrace{131, 131, \ldots, 131}_{7}, 132),$$

has h^*-vector

$$h^*(\Delta_w) = (1, \underbrace{0, 0, \ldots, 0}_{7}, 131, \underbrace{0, 0, \ldots, 0}_{7}).$$

What about the unimodality of f^*-vectors?

Nonunimodal f^*-vectors

Example

The simplex of dimension 15 such that

$$\Delta_w = \operatorname{conv}\left\{0, e_1, e_2, ..., e_{14}, w\right\},$$

$$\text{where} \quad w = (\underbrace{1, 1, \dots, 1}_{7}, \underbrace{131, 131, \dots, 131}_{7}, 132),$$

has f^*-vector

$$\begin{aligned} f^*(\Delta_w) = &(1, 16, 120, 560, 1820, 4368, 8008, 11440, 13001, \\ &12488, 11676, 11704, 10990, 7896, 3788, 1064, 132). \end{aligned}$$

In particular, $f^*_8 \geq f^*_9 \leq f^*_{10} \geq f^*_{11}$.

So far, this is the smallest-dimensional example we have found.

- Hence f^*-vectors are not unimodal in general!

Happens in the best vector families...

Let P be a d-dimensional polytope that is **simplicial**, i.e., all its faces are simplices. Björner showed that $f(P)$ is not unimodal for all P, but ...

Theorem (Björner,1981)

The f-vector of a simplicial d-polytope P with $d \geq 3$ satisfies

$$f_{-1} < f_0 < f_1 < \cdots < f_{\lfloor \frac{d}{2} \rfloor - 1} \leq f_{\lfloor \frac{d}{2} \rfloor} \quad \textit{and} \quad f_{\lfloor \frac{3(d-1)}{4} \rfloor - 1} > \cdots > f_{d-1}.$$

(Björner,1994): In fact, for p with $\lfloor \frac{d}{2} \rfloor \leq p \leq \lfloor \frac{3(d-1)}{4} \rfloor$, there is a simplicial d-polytope whose f-vector is unimodal with a peak at p:

$$f_{-1} < f_0 < f_1 < \cdots < f_{p-1} < f_p > f_{p+1} > \cdots > f_{d-1}.$$

Shape of f-vectors

Theorem (Björner,1986)

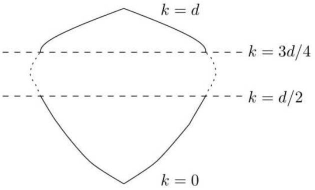
Moreover, $f(P) = (f_0, f_1, \ldots, f_k, \ldots, f_{\lfloor \frac{d}{2} \rfloor}, \ldots, f_{d-2-k}, f_{d-1-k}, \ldots, f_{d-1})$ *satisfies*

$$f_k < f_{d-2-k},$$

$$f_k \leq f_{d-1-k},$$

for $0 \leq k \leq \lfloor \frac{(d-3)}{2} \rfloor$.

Thus, this is roughly how the "shape" of the **face lattice** of simplicial polytopes looks like.



(G. Ziegler, Lectures on Polytopes, Springer, 1995)

Question: Are there analogous inequalities for f^*-vectors?

Theorem (BDHV)

The f^*-vector of a d-dimensional lattice polytope, $d \geq 2$, satisfies

$$f^*_{-1} < f^*_0 < f^*_1 < \cdots < f^*_{\lfloor \frac{d}{2} \rfloor - 1} \leq f^*_{\lfloor \frac{d}{2} \rfloor} \quad \textit{and} \quad f^*_{\lfloor \frac{3d-1}{4} \rfloor} > \cdots > f^*_d .$$

Moreover, for the bounds $\lfloor \frac{d}{2} \rfloor$ and $\lfloor \frac{3d-1}{4} \rfloor$ we have:

- If P is the d-dimensional unimodular simplex Δ then

$$f^*_{-1} < f^*_0 < \cdots < f^*_{\lfloor \frac{d}{2} \rfloor - 1} \leq f^*_{\lfloor \frac{d}{2} \rfloor} > f^*_{\lfloor \frac{d}{2} \rfloor + 1} > \cdots > f^*_d .$$

- If P is the d-dimensional cube $[-1, 1]^d$ then

$$f^*_{-1} < f^*_0 < \cdots < f^*_{\lfloor \frac{3d-1}{4} \rfloor} > \cdots > f^*_d$$

holds (at least) for $d \leq 9$.

Theorem (BDHV)

The f^*-vector of a d-dimensional lattice polytope, $d \geq 2$, satisfies

$$f^*_{-1} < f^*_0 < f^*_1 < \cdots < f^*_{\lfloor \frac{d}{2} \rfloor - 1} \leq f^*_{\lfloor \frac{d}{2} \rfloor} \quad \textit{and} \quad f^*_{\lfloor \frac{3d-1}{4} \rfloor} > \cdots > f^*_d.$$

Comment: When P is the unimodular simplex, the theorem holds by Björner's inequalities for f-vectors.

- The proof makes use of the relation between f^* and h^* vectors, and the inequalities given by Hibi:

$$h^*_d + h^*_{d-1} + \cdots + h^*_{d-k} \ \leq \ h^*_0 + h^*_1 + \cdots + h^*_{k+1}$$

for $k = 0, \ldots, \left\lfloor \frac{d}{2} \right\rfloor - 1$, for a d-dimensional lattice polytope P.

Definition

We say that the d-dimensional lattice polytope P is **Gorenstein of index** g, $g \geq 1$, whenever the polynomial $h^*(P;z) := \sum_{k=0}^{d} h_k^* z$ has degree $d+1-g$ and is symmetric with respect to its degree.

Example: The 5-dimensional unit cube $P = [0,1]^5$ with h^*-vector

$$h^*(P) = (1, 26, 66, 26, 1, 0)$$

is Gorenstein of index 2.

Theorem (BDHV)

Let P be a d-dimensional Gorenstein polytope of index g. Then

$$f_k^* > \cdots > f^*_{\lfloor \frac{3d-1}{4} \rfloor} > \cdots > f_d^* \quad \textit{for} \quad k = \frac{1}{2}\left(d - 1 + \left\lfloor \frac{d+1-g}{2} \right\rfloor\right).$$

Proposition (BDHV)

Let P be a d-dimensional lattice polytope. Then,

$$f_k^* < f_{d-2-k}^* \quad \text{and} \quad f_k^* \leq f_{d-1-k}^*, \quad \text{for} \quad 0 \leq k \leq \frac{(d-3)}{2}.$$

Moreover, if $h_d^* \neq 0$ and $h^*(P) \neq (1,1,\ldots,1)$ then

$$f_k^* < f_{d-k}^*, \quad \text{for} \quad 0 < k < \frac{d}{2},$$
$$\text{and} \quad f_0^* \leq f_d^*.$$

Note: It follows that for every d-dimensional lattice polytope:

$$\min\{f_0^*, f_d^*\} \leq f_k^*, \quad \text{for} \;\; 0 \leq k \leq d.$$

The analogous question for f-vectors is harder. Bárány asked it in 1997 and it was only recently answered positively:

$$\min\{f_0, f_{d-1}\} \leq f_k, \quad \text{for} \;\; 0 \leq k \leq d-1 \;\; \text{(Hinman, 2022+)}.$$

Corollary: It directly follows from the theorem that polytopes of dimension $2 \leq d \leq 6$ have unimodal f^*-vectors. In fact:

Proposition (BDHV)

The f^*-vector of a d-dimensional lattice polytope, where $1 \leq d \leq 10$, is unimodal.

Another family of unimodal f^*-vectors is the following:

Proposition (BDHV)

Let P be a d-dimensional lattice polytope such that

$$h_k^* = 0 \text{ for } k \geq 4.$$

Then $f^*(P)$ is unimodal with a peak either at $f^*_{\lfloor \frac{d}{2} \rfloor}$ or $f^*_{\lfloor \frac{d}{2} \rfloor + 1}$.

Future work:

- Compute f^*-vectors for other families of polytopes.
- Is $f^*(P)$ unimodal when P admits a unimodular triangulation?
- Are there polytopes with unimodal h^*-vector and nonunimodal f^*-vector?
- So far we know that for a lattice d-dimensional polytope P, $f^*(P)$ is unimodal when $d \leq 10$, but unimodality fails for polytopes of order 15. Can we close this gap for $11 \leq d \leq 14$?

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