Geometry & Combinatorics meet zonoids

joint work with Fulvio Gesmundo

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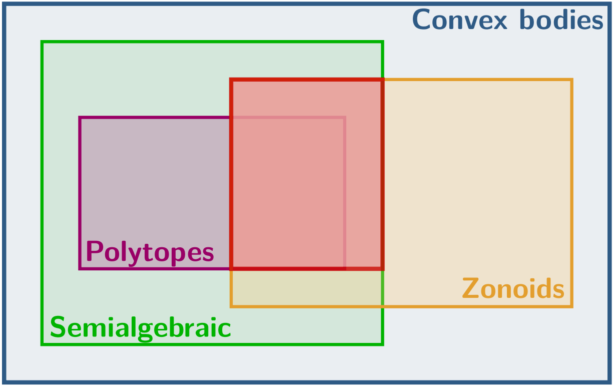
MAX-PLANCK-GESELLSCHAFT

The world of convex bodies



DISCOTOPES!

→ How to recognise a zonoid?

hard: $\{\text{zonoids}\} \subsetneq \{\text{centrally symmetric convex bodies}\}$
[Schneider, Weil, ...]

→ How to recognise a semialgebraic zonoid?

still hard [Lerario, Mathis]

→ How to recognise a discotope?

→ How to recognise a zonotope?

easy: check if its 2-dimensional faces are centrally symmetric
[Bolker, Schneider, ...]

Previous work

Karim A. Adiprasito and Raman Sanyal,
Whitney numbers of arrangements via measure concentration of intrinsic volumes,
arXiv:1606.09412

Léo Mathis and C.M.,
Fiber Convex Bodies,
arXiv:2105.12406, to appear in Discrete & Computational Geometry

Setting and definition

Disc = linear image of the unit ball of $\mathbb{R}^n$ in $\mathbb{R}^d$ $(n \leq d)$.

$\rightsquigarrow$ they are semialgebraic zonoids

Definition

Consider the discs $D_1, \ldots, D_N \subset \mathbb{R}^d$. The associated discotope is the Minkowski sum

$$\mathcal{D} = D_1 + \ldots + D_N.$$

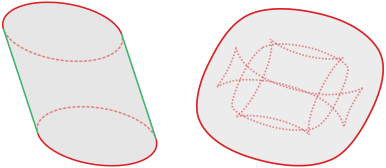
Denote by N_m the number of discs of dimension m; we say that $\mathcal{D}$ is of *type* $\mathbf{N} = (N_1, \ldots, N_d)$. Notice: $N = \sum_{m=1}^{N} N_m$.

$\rightsquigarrow$ they are semialgebraic zonoids

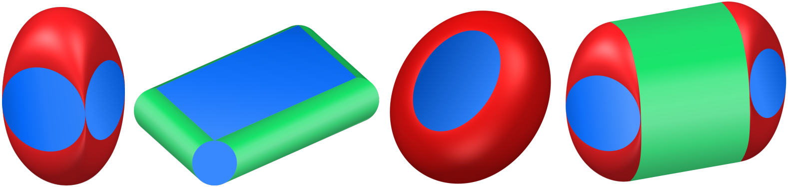
Our goal: characterize *generic* discotopes according to their type.

Gallery

$d = 2$:

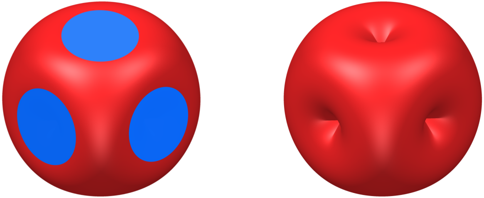


$d = 3$:



The purely nonlinear part

Our goal: study the exposed points *algebraically*.
$\rightsquigarrow \mathrm{Ex}(\mathcal{D})$ is the Zariski closure in $\mathbb{C}^d$ of the set of exposed points.



Consider the addition map $\Sigma : \partial D_1 \times \ldots \times \partial D_N \to \mathbb{R}^d \subset \mathbb{C}^d$ and define

$$\mathcal{S} = \overline{\mathrm{im}(\Sigma) \cap \partial \mathcal{D}} \subset \mathbb{C}^d$$

the purely nonlinear part of $\mathcal{D}$. Then $\mathrm{Ex}(\mathcal{D}) \subseteq \mathcal{S}$.

Some results on $\mathcal{S}$

Let $\mathcal{D}$ be a discotope of type $\mathbf{N} = (0, N_2, \ldots, N_d)$ and consider

$$(\star) = \sum_{m=1}^{d} (m-1) N_m$$

Theorem (Gesmundo-M. 2022)

- if $(\star) \leq d-1$ then $\mathcal{S}$ is an irreducible variety with

$$\dim \mathcal{S} = (\star) \text{ and } \deg \mathcal{S} = 2^N,$$

- if $(\star) \geq d-1$ then $\dim \mathcal{S} = d-1$.

Degree and irreducibility in the case $(\star) > d-1$?

General case

Conjecture

$\mathcal{S}$ is irreducible.

This would imply that actually $\mathcal{S} = \mathrm{Ex}(\mathcal{D})$.

Introduce another variety: the critical locus of the addition map Σ.

General fact: $\Sigma^{-1}(\mathcal{S}) \subseteq \operatorname{crit}\Sigma$

Idea: we conjecture that already the critical locus of the addition map is irreducible. This would imply that $\mathcal{S}$ is irreducible as well.

Let $\mathcal{D} = D_1 + \ldots + D_N$ where $\dim D_i = 2$ for every i.

Theorem (Gesmundo-M. 2022)

The variety $\operatorname{crit} \Sigma$ is irreducible, of dimension $d-1$ and degree $2^N \cdot \binom{N}{d-1}$.

Idea of the proof.

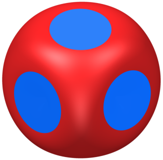
Adapt Bertini's Theorem to specific non-generic (but generic enough) linear cuts of a determinantal variety. □

Corollary

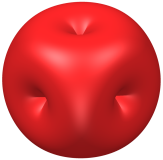
The variety $\mathcal{S}$ is irreducible, of dimension $d-1$ and degree $\deg \mathcal{S} \leq 2^N \cdot \binom{N}{d-1}$.

A case of study: the dice

Consider the dice $\mathcal{D} = D_1 + D_2 + D_3 \subset \mathbb{R}^3$, where



$$D_1 = \{(x_1, x_2, x_3) : x_1 = 0; x_2^2 + x_3^2 \leq 1\},$$
$$D_2 = \{(x_1, x_2, x_3) : x_2 = 0; x_1^2 + x_3^2 \leq 1\},$$
$$D_3 = \{(x_1, x_2, x_3) : x_3 = 0; x_1^2 + x_2^2 \leq 1\}.$$



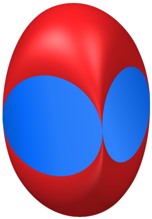
Its purely nonlinear part $\mathcal{S} = \mathrm{Ex}\,\mathcal{D}$ is an irreducible surface of degree $24 = 2^3 \cdot \binom{3}{2}$.

Theorem (Gesmundo-M. 2022)

The surface $\mathcal{S}$ is birational to a K3 surface. Explicitly, a desingularization of $\mathcal{S}$ is the variety of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined by

$$(y_1^2 - z_1^2)(y_2^2 - z_2^2)(y_3^2 - z_3^2) - 8y_1y_2y_3z_1z_2z_3 = 0.$$

Fix a generic discotope $\mathcal{D} \subset \mathbb{R}^d$ and let $L_i = \langle D_i \rangle$, so that $L_1, \ldots, L_N$ are N generic linear subspaces of $\mathbb{R}^d$. Consider a hyperplane H transversal to the L_i.



A point $p \in \partial\mathcal{D}$ has a normal cone of dimension bigger than one if and only if

$$\dim \Big\langle (H \cap L_1), \ldots, (H \cap L_N) \Big\rangle < d - 1.$$

Question

When can such H exist? Are there conditions on $\dim L_i$?

Thank you!

Fulvio Gesmundo and C.M.,
The Geometry of Discotopes,
Le Matematiche, **77**(1), 143–171 (2022)