Geometry & Combinatorics meet zoonoids

joint work with Fulvio Gesmundo

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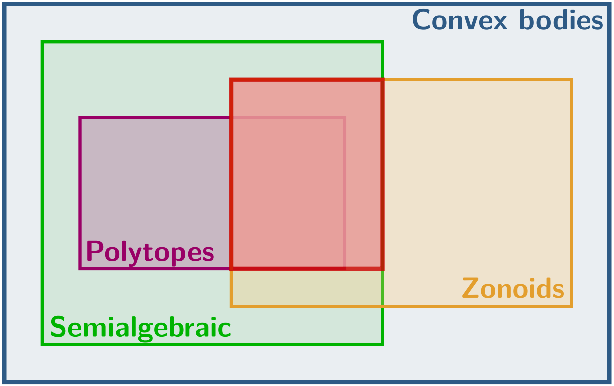
MAX-PLANCK-GESELLSCHAFT

The world of convex bodies



DISCOTOPES!

→ How to recognise a zonoid?

hard: $\{$zonoids$\} \subsetneq \{$centrally symmetric convex bodies$\}$
[Schneider, Weil, . . .]

→ How to recognise a semialgebraic zonoid?

still hard [Lerario, Mathis]

→ How to recognise a discotope?

→ How to recognise a zonotope?

easy: check if its 2-dimensional faces are centrally symmetric
[Bolker, Schneider, . . .]

Previous work

Karim A. Adiprasito and Raman Sanyal,
Whitney numbers of arrangements via measure concentration of intrinsic volumes,
arXiv:1606.09412

Léo Mathis and C.M.,
Fiber Convex Bodies,
arXiv:2105.12406, to appear in Discrete & Computational Geometry

Setting and definition

Disc = linear image of the unit ball of $\mathbb{R}^n$ in $\mathbb{R}^d$ ($n \leq d$).

they are semialgebraic zonoids

Definition

Consider the discs $D_1, \ldots, D_N \subset \mathbb{R}^d$. The associated discotope is the Minkowski sum

$$D = D_1 + \ldots + D_N .$$

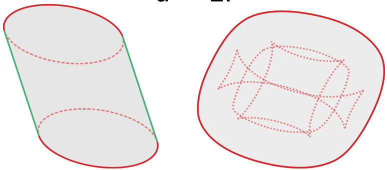
Denote by N_m the number of discs of dimension m; we say that D is of *type* $\mathtt{N} = (N_1, \ldots, N_d)$. Notice: $N = \sum_{m=1}^{N} N_m$.

they are semialgebraic zonoids

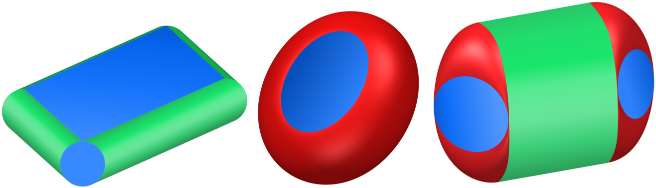
Our goal: characterize *generic* discotopes according to their type.

Gallery

$d = 2$:



$d = 3$:



The purely nonlinear part

Our goal: study the exposed points *algebraically*.

$\mathrm{Ex}(D)$ is the Zariski closure in $\mathbb{C}^d$ of the set of exposed points.

Consider the addition map $\Sigma : \partial D_1 \times \ldots \times \partial D_N \to \mathbb{R}^d \quad \mathbb{C}^d$
and define

$$S = \overline{\mathrm{im}(\Sigma) \cap \partial D} \quad \mathbb{C}^d$$

the purely nonlinear part of D. Then $\mathrm{Ex}(D) \quad S$.

Some results on S

Let D be a discotope of type $\mathtt{N} = (0, N_2, \ldots, N_d)$ and consider

$$(\mathrm{F}) = \sum_{m=1}^{d} (m-1) N_m$$

Theorem (Gesmundo-M. 2022)

- if $(\mathrm{F}) \leq d-1$ then S is an irreducible variety with
$$\dim S = (\mathrm{F}) \text{ and } \deg S = 2^N,$$
- if $(\mathrm{F}) \geq d-1$ then $\dim S = d-1$.

Degree and irreducibility in the case $(\mathrm{F}) > d-1$?

Conjecture

S is irreducible.

This would imply that actually $S = \mathrm{Ex}(D)$.

Introduce another variety: the critical locus of the addition map Σ.

General fact: $\Sigma^{-1}(S) \quad \mathrm{crit}\,\Sigma$

Idea: we conjecture that already the critical locus of the addition map is irreducible. This would imply that S is irreducible as well.

Type $(0, N, 0, \ldots, 0)$ with $N \geq d$

Let $D = D_1 + \ldots + D_N$ where $\dim D_i = 2$ for every i.

Theorem (Gesmundo-M. 2022)

The variety $\operatorname{crit} \Sigma$ is irreducible, of dimension $d - 1$ and degree $2^N \cdot \binom{N}{d-1}$.

Idea of the proof.

Adapt Bertini's Theorem to specific non-generic (but generic enough) linear cuts of a determinantal variety. □

Corollary

The variety S is irreducible, of dimension $d - 1$ and degree $\deg S \leq 2^N \cdot \binom{N}{d-1}$.

A case of study: the dice

Consider the dice $D = D_1 + D_2 + D_3 \subset \mathbb{R}^3$, where

$$D_1 = \{(x_1, x_2, x_3) : x_1 = 0; x_2^2 + x_3^2 \leq 1\},$$
$$D_2 = \{(x_1, x_2, x_3) : x_2 = 0; x_1^2 + x_3^2 \leq 1\},$$
$$D_3 = \{(x_1, x_2, x_3) : x_3 = 0; x_1^2 + x_2^2 \leq 1\}.$$

Its purely nonlinear part $S = \mathrm{Ex}\, D$ is an irreducible surface of degree $24 = 2^3 \cdot \binom{3}{2}$.

Theorem (Gesmundo-M. 2022)

The surface S is birational to a K3 surface. Explicitly, a desingularization of S is the variety of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined by

$$(y_1^2 - z_1^2)(y_2^2 - z_2^2)(y_3^2 - z_3^2) - 8 y_1 y_2 y_3 z_1 z_2 z_3 = 0.$$

Open Problem 〉 Polymatroids?

Fix a generic discotope $D \quad \mathbb{R}^d$ and let $L_i = \; D_i$, so that $L_1, \ldots, L_N$ are N generic linear subspaces of $\mathbb{R}^d$. Consider a hyperplane H transversal to the L_i.

A point $p \quad \partial D$ has a normal cone of dimension bigger than one if and only if

$$\dim \; (H \cap L_1), \ldots, (H \cap L_N) \; < d - 1.$$

Question

When can such H exist? Are there conditions on $\dim L_i$?

Fulvio Gesmundo and C.M.,
The Geometry of Discotopes,
Le Matematiche, **77**(1), 143–171 (2022)