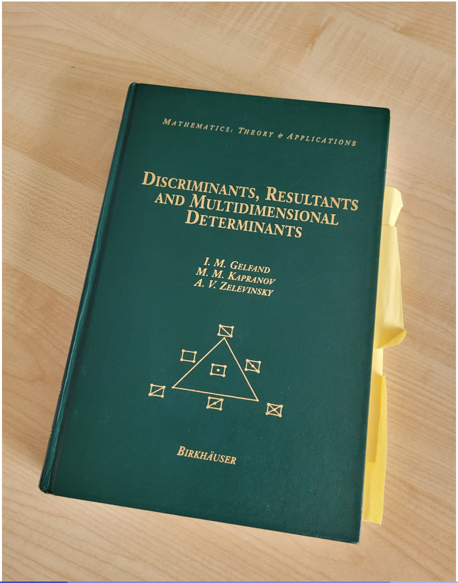
What are thin polytopes?

Benjamin Nill
OVGU Magdeburg

Geometry meets Combinatorics, Bielefeld, 7.9.2022

1. GKZ - thin simplices



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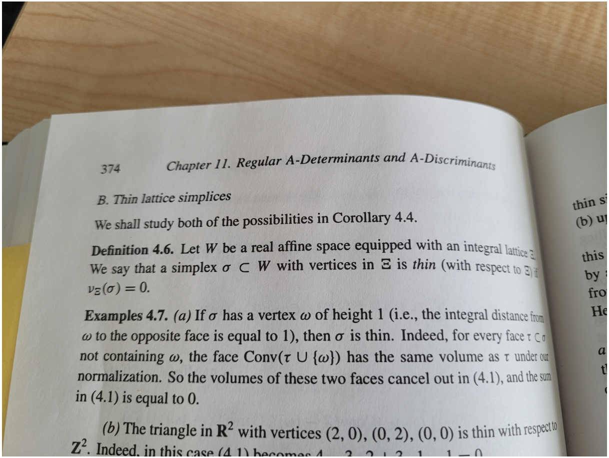

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B. Thin lattice simplices

We shall study both of the possibilities in Corollary 4.4.

Definition 4.6. Let W be a real affine space equipped with an integral lattice Ξ. We say that a simplex $\sigma \subset W$ with vertices in Ξ is *thin* (with respect to Ξ) if $\nu_\Xi(\sigma) = 0$.

Examples 4.7. *(a)* If σ has a vertex ω of height 1 (i.e., the integral distance from ω to the opposite face is equal to 1), then σ is thin. Indeed, for every face $\tau \subset \sigma$ not containing ω, the face $\mathrm{Conv}(\tau \cup \{\omega\})$ has the same volume as τ under our normalization. So the volumes of these two faces cancel out in (4.1), and the sum in (4.1) is equal to 0.

(b) The triangle in $\mathbf{R}^2$ with vertices $(2, 0)$, $(0, 2)$, $(0, 0)$ is thin with respect to $\mathbf{Z}^2$. Indeed, in this case (4.1) becomes ...

1. GKZ - thin simplices

Normalized volume

Let $P \subset \mathbb{R}^d$ be a **lattice polytope**: polytope with vertices in $\mathbb{Z}^d$.

The **normalized volume** $\mathrm{vol}_{\mathbb{Z}}(P)$ is defined such that

$$\mathrm{vol}_{\mathbb{Z}}(P) = 1$$

$$\Longleftrightarrow$$

P is the convex hull of an affine lattice basis of $\mathbb{Z}^d \cap \mathrm{aff}(P)$.

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- If P has dimension d, then $\mathrm{vol}_{\mathbb{Z}}(P) = d!\,\mathrm{vol}(P)$.
- $\mathrm{vol}_{\mathbb{Z}}(P) \in \mathbb{Z}_{\geq 1}$.
- Special cases: $\mathrm{vol}_{\mathbb{Z}}(\{\text{point}\}) = 1$; $\mathrm{vol}_{\mathbb{Z}}(\emptyset) := 1$.

1. GKZ - thin simplices

Let $S \subset \mathbb{R}^d$ be d-dimensional **lattice simplex**.

Thin simplices (GKZ '94)

S is **thin** if its **Newton number** vanishes:

$$\nu(S) := \sum_{F \in [\emptyset, S]} (-1)^{\dim(S) - \dim(F)} \operatorname{vol}_{\mathbb{Z}}(F) = 0,$$

where $[\emptyset, S]$ is the face poset of S.

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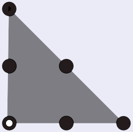
Example



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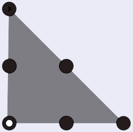

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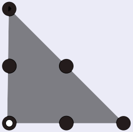

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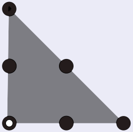

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S is called **hollow** if S has no lattice points in its relative interior.

Theorem (GKZ '94)

Let $d \geq 1$.

- $\nu(S) \geq 0$.
- S lattice pyramid $\implies$ S thin $\implies$ S hollow.
- For $d = 1$ or $d = 2$: S thin $\iff$ S hollow.

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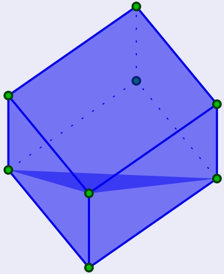
(GKZ '94) *"A classification of thin lattice simplices seems to be an interesting problem in the geometry of numbers."*

1. GKZ - thin simplices

Combinatorial viewpoint

$\mathrm{vol}_{\mathbb{Z}}(S)$ equals number of lattice points in *half-open* parallelepiped $\Pi(S)$ spanned by $S \times \{1\}$.



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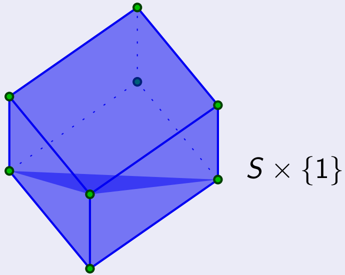

$\nu(S)$ equals number of **interior** lattice points in $\Pi(S)$.

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Questions:

1. Classification known for $d = 2$. *What about* $d = 3$*?*

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1. Classification known for $d = 2$. *What about* $d = 3$*?*
2. Formula sometimes **negative** for polytopes. *What are thin polytopes*?

2. Stanley - local h^*-polynomials

Background

(Stanley '87): Definition of **(toric)** g**- and** h**-polynomials** of (lower) Eulerian posets. For P rational polytope, $g_{[\emptyset,P)}(t)$ has nonnegative coefficients; $h_{[\emptyset,P)}(t)$ has nonnegative, palindromic, unimodal coefficients.

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(Stanley '92): h-vectors of triangulations of a polytope are monotone under refinements.

Main tool: Nonnegativity, palindromicity [and unimodality] of **local** h-**polynomials** $l_{\mathcal{T}}(t)$ of [regular] subdivisions $\mathcal{T}$ of polytopes.

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Let $P \subset \mathbb{R}^d$ be d-dimensional **lattice polytope**.
P has h^***-polynomial** $h^*_P(t) \in \mathbb{Z}_{\geq 0}[t]$:

$$1 + \sum_{n \geq 1} |(nP) \cap \mathbb{Z}^d| t^n = \frac{h^*_P(t)}{(1-t)^{d+1}}$$

with

$$h^*_P(1) = \mathrm{vol}_{\mathbb{Z}}(P).$$

2. Stanley - local h^*-polynomials

Definition (Stanley '92; Borisov-Mavlyutov '03; Katz-Stapledon '16)

P lattice polytope has **local h^*-polynomial**:

$$l_P^*(t) := \sum_{F \in [\emptyset, S]} (-1)^{\dim(P) - \dim(F)} \, h_F^*(t) \, g_{(F,P]^*}(t).$$

Proposition: It is a **palindromic** polynomial (reflected at $(d+1)/2$).

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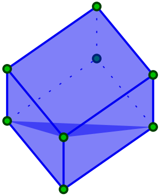

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If $P = S$ is a *simplex*, then $g_{(F,S]^*}(t) = 1$. Thus:

$$l^*_S(1) = \nu(S),$$

$l^*_S(t)$ is the **box polynomial**:
ith coefficient of $l^*_S(t)$ counts *interior* lattice points in $\Pi(S)$ on height i.

2. Stanley - local h^*-polynomials

Theorem (Karu '06; Borisov-Mavlyutov '03)

$l_P^*(t)$ has **nonnegative** coefficients.

(Batyrev, Borisov, Mavlyutov, Schepers): $l_P^*(t)$ (called $\tilde{S}$-**polynomial**) appears naturally in computing the Hodge-Deligne polynomial (E-polynomial) of generic hypersurface (and complete intersections) in toric varieties.

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$$"\, l^* = h^* * g^{-1}" \quad \rightsquigarrow \quad "\, h^* = l^* * g"$$

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Proposition (Stanley '92; ...)

$$h^*_P(t) = \sum_{F \in [\emptyset, P]} l^*_F(t) g_{[F,P)}(t) = g_{[\emptyset,P)} + \sum_{F \in (\emptyset, P)} l^*_F(t) g_{[F,P)}(t) + l^*_P(t).$$

In particular, $l^*_P(t) \leq h^*_P(t)$ coefficientwise.

$l^*_P(t)$ should be seen as the 'core' of the h^*-polynomial.

3. Borger-Kretschmer-N. - thin polytopes

Definition

P lattice polytope is **thin** if $l_P^*(t) = 0$, or equivalently, $l_P^*(1) = 0$.

Proposition

Let $d \geq 1$.

- P lattice pyramid $\implies$ P thin $\implies$ P hollow.
- For $d = 1$ or $d = 2$: P thin $\iff$ P hollow.

Classification of thin polytopes known for $d \leq 2$.

3. Borger-Kretschmer-N. - thin polytopes

Theorem (Borger-Kretschmer-N. '22)

Let $d = 3$. Then P is thin if and only if

1. $h^*_P(t)$ has degree at most one, or
2. P is a lattice pyramid.

Corollary

All thin lattice tetrahedra are lattice pyramids.

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Proof uses classification of 3-dimensional hollow lattice polytopes (Averkov, Wagner, Weismantel '10) and

$$l^*_P(t) = |\text{int}(P) \cap \mathbb{Z}^3| \cdot t +$$

$$\left(|\text{int}(2P) \cap \mathbb{Z}^3| - 4\,|\text{int}(P) \cap \mathbb{Z}^3| - \sum_{F \leq P \text{ facet}} |\text{int}(F) \cap \mathbb{Z}^3| \right) \cdot t^2 +$$

$$|\text{int}(P) \cap \mathbb{Z}^3| \cdot t^3.$$

3. Borger-Kretschmer-N. - thin polytopes

Theorem (Borger-Kretschmer-N. '22)

Let $d = 3$. Then P is thin if and only if

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2. P is a lattice pyramid.

What about higher dimensions?

Constructing thin polytopes

1. P is **trivially thin** if $h_P^*(t)$ has degree at most $\frac{d}{2}$.
2. P is thin if it is a $\mathbb{Z}$-**join** of P_1 and P_2 with P_1 thin.

For $d \geq 4$, not all thin polytopes are given this way - but true generically?

3. Borger-Kretschmer-N. - thin polytopes

P is called **spanning** if its lattice points affinely generate $\mathbb{Z}^d$.

Motivating question

Is every spanning thin lattice polytope of types (1) or (2)?

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Theorem (N.-Schepers '12; Borger-Kretschmer-N. '22)

Answer is YES for P **Gorenstein** (i.e., $h^*_P(t)$ is palindromic).
Moreover:

- Thinness is invariant under duality of Gorenstein polytopes.
- Thin Gorenstein polytopes have lattice width 1.
- Thin Gorenstein simplices are lattice pyramids.

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Proof needs general crucial fact:
Thinness stays invariant under coarsening the lattice.

4. Katz-Stapledon - decomposing l^*

Let $\mathcal{T}$ be triangulation of P into lattice simplices.

Theorem (Betke-McMullen '85)

$$h_P^*(t) = \sum_{S \in \mathcal{T}} l_S^*(t)\, h_{\mathrm{link}(\mathcal{T},S)}(t).$$

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where *relative local h-polynomial* $l_{\mathcal{T},S}(t)$ has **nonnegative** coefficients.

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Consequences

- Thinness stays invariant under coarsening the lattice.
- If $\mathcal{T}$ is unimodular triangulation, then $l^*_P = l_{\mathcal{T},\emptyset} = l_{\mathcal{T}}$ is *unimodal* (Stanley '92).