MANY POLYTOPES

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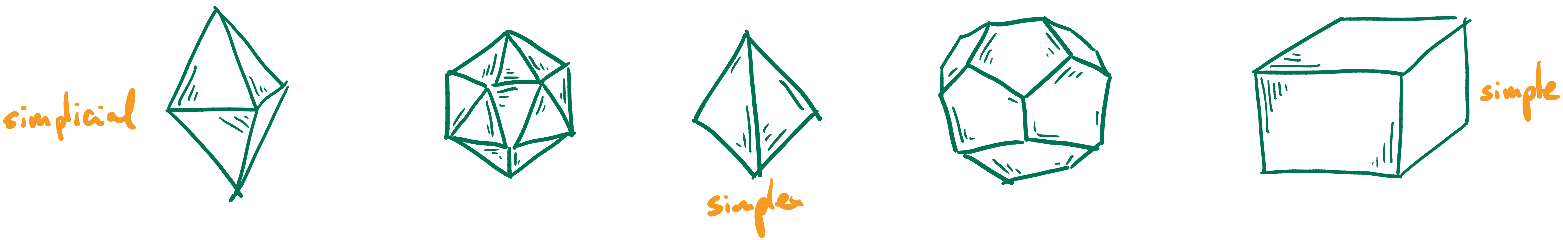
Geometry meets Combinatorics in Bielefeld
5 / 9 / 2022

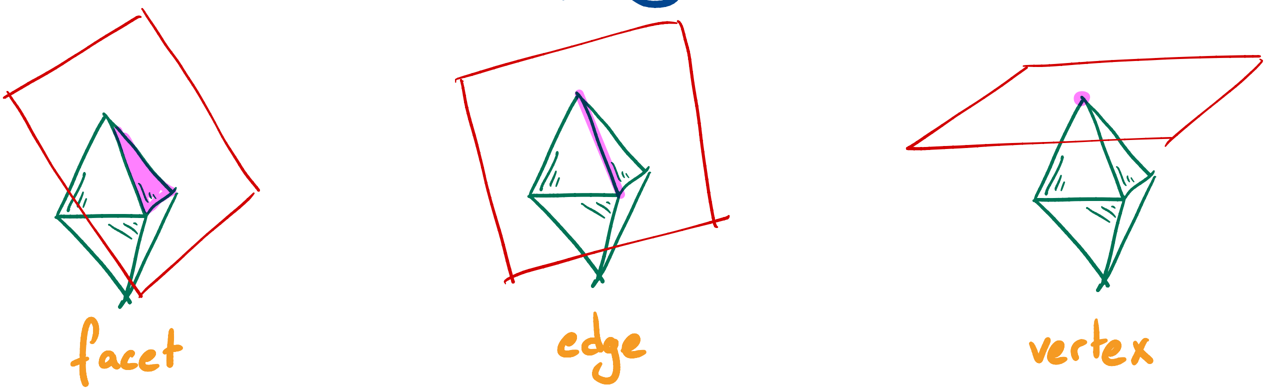
Polytope: convex hull of finitely many vertices

$\parallel\!\mid$

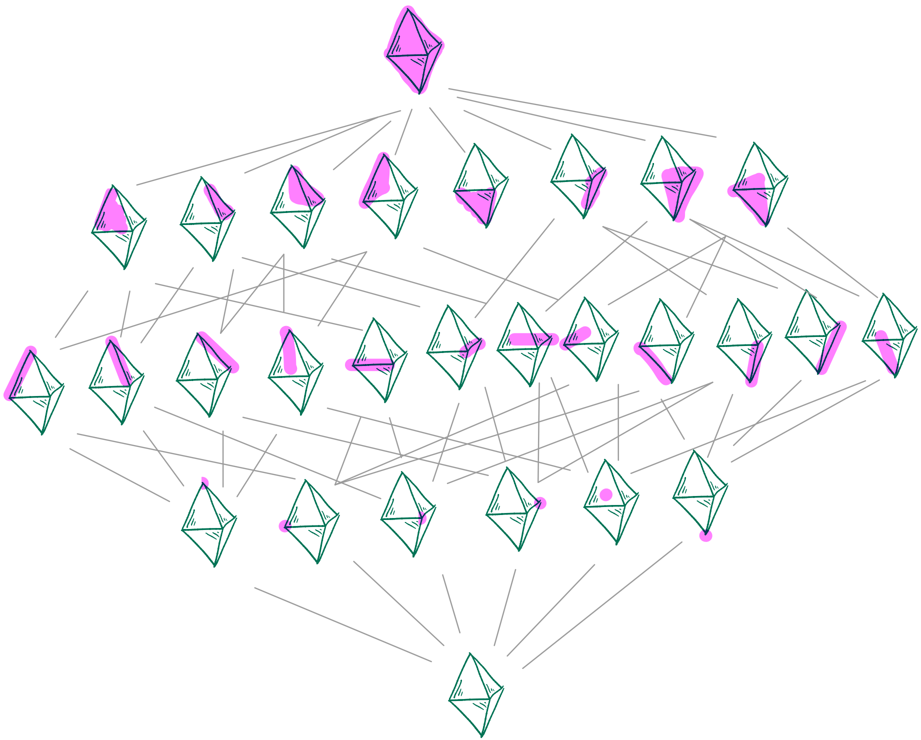
bounded intersection of finitely many halfspaces



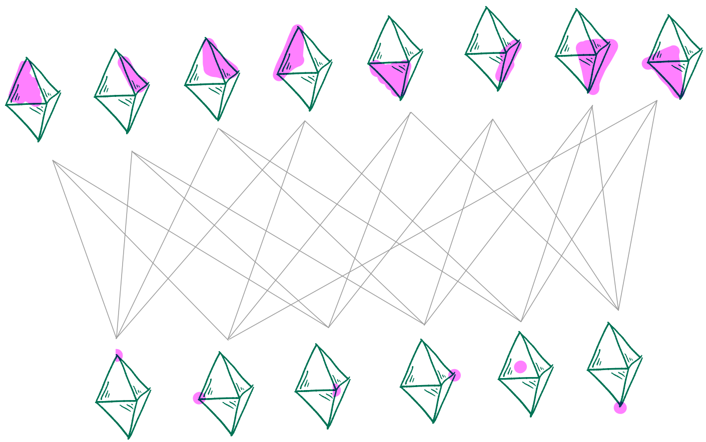
Face: intersection with supporting halfspace



Face lattice:

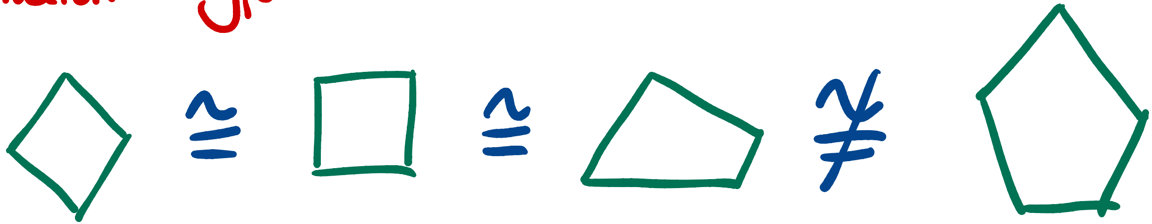


Face lattice:

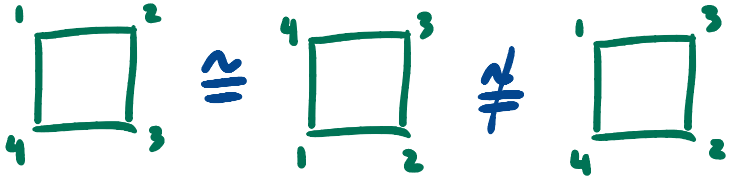


Determined by vertex-facet incidences

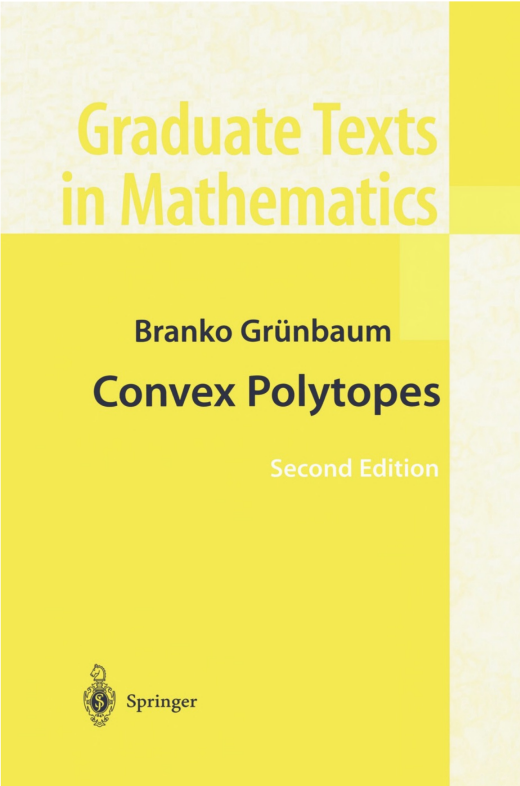
Combinatorial type:



Labeled combinatorial type:



What is the number of (labeled) combinatorial types of d-polytopes with n vertices?



Preface

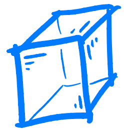
[...]

About the turn of the century, however, a steep decline in the interest in convex polytopes was produced by two causes working in the same direction. Efforts at enumerating the different combinatorial types of polytopes, started by Euler and pursued with much patience and ingenuity during the second half of the XIX$^{\text{th}}$ century, failed to produce any significant results even in the three-dimensional case; this lead to a widespread feeling that the interesting problems concerning polytopes are hopelessly hard. Simultaneously, the ascendance of Klein's "Erlanger Program" and the spread of its normative influence tended to cast the preoccupation with the combinatorial theory of convex polytopes into a rather disreputable rôle—and that at a time when such "legitimate" fields as algebraic geometry and in particular topology started their spectacular development.

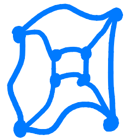
[...]

$\boxed{d = 3}$

Thm: [Steinitz 1922]

G is the graph of a 3-polytope  geometric condition

$\Updownarrow$ $\updownarrow$

G is 3-connected & planar  combinatorial/topological condition

Thm: [Whitney 1932]

The face lattice of a 3-polytope is determined by its graph

Thm: [Tutte 1962]

The number of rooted simplicial 3-polytopes with $n+1$ vertices is:

$$= \frac{2(4n-11)!}{(3n-7)!\,(n-2)!} \approx \frac{3}{16\sqrt{6\pi n^5}} \left(\frac{256}{27}\right)^{n-2}$$

Thm: [Wormald-Bender 1988]

The number of rooted 3-polytopes with $n+1$ vertices & $m+1$ facets is: $= \left(\cdots \right) \sim \frac{1}{3^5\, n\, m} \binom{2m}{n+3} \binom{2n}{m+3}$

& the number of combinatorial types of 3-polytopes with $n+1$ vertices & $m+1$ facets is $\sim \frac{1}{2^2\, 3^5\, n\, m\, (n+m)} \binom{2m}{n+3} \binom{2n}{m+3}$

$\boxed{d \geq 4}$ Universality

Realization space: P d-polytope n vertices

$$\mathcal{R}(P) = \{ Q \text{ polytope} \cong P \} \subseteq \mathbb{R}^{nd}$$

parametrized by vertex coordinates

<u>Thm:</u> [Mnëv 1988 ; Richter-Gebert 1995]
$\forall$ primary basic semialgebraic set S
$\exists$ 4-polytope P whose realization space is stably equivalent to S.

$$\boxed{\text{It is hard } (\exists\mathbb{R}\text{-hard} \Rightarrow \text{NP-hard}) \text{ to decide polytopality of a "face lattice"}}$$

Upper Bound Theorem: [McMullen 1970, Stanley 1975]

The number of facets of a d-polytope (or a simplicial $(d-1)$-sphere) with n vertices is $O(n^{\lfloor d/2 \rfloor})$

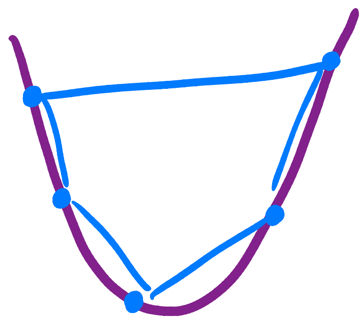
The upper bound is attained by simplicial neighborly polytopes/spheres.

↳ Every subset of $\leq \lfloor d/2 \rfloor$ vertices forms a face.

e.g. Cyclic polytopes

$\gamma : t \mapsto (t, t^2, t^3, \ldots, t^d)$

$C_d(n) = \mathrm{conv}(\gamma(t_1), \ldots, \gamma(t_n))$



Corollary:

The number of simplicial $(d-1)$-spheres with n vertices is

$$\leq \binom{n}{d}^{O(n^{\lfloor d/2 \rfloor})} = n^{O(n^{\lfloor d/2 \rfloor})}$$

Prop: [Kalai 1988]

The number of simplicial $(d-1)$-spheres with n vertices is

$$\leq n^{2n^{\lfloor d/2 \rfloor}(1+o(1))}$$

Many simplicial spheres

Thm: [Kalai 1988, Nevo-Santos-Wilson 2016]

The number of simplicial $(d-1)$-spheres with n vertices is

$$\geq 2^{\Omega\left(n^{\lfloor d/2 \rfloor}\right)}$$

Kalai

$$\geq 2^{\frac{1}{d+1} n^{\lfloor (d-1)/2 \rfloor}(1+o(1))}$$

Nevo-Santos-Wilson

d even

$$\geq 2^{\frac{2}{3d^{d+1}} n^{d/2}(1+o(1))}$$

Summing up:

$$2^{\Omega(n^{\lfloor d/2 \rfloor})} \leq \begin{array}{c}\#\text{ simplicial} \\ (d-1)\text{-spheres} \\ \text{with } n \text{ vertices}\end{array} \leq n^{O(n^{\lfloor d/2 \rfloor})}$$

$$\begin{array}{c} \rotatebox{90}{$\leq$} \\ \#\text{ simplicial} \\ d\text{-polytopes} \\ \text{with } n \text{ vertices}\end{array}$$

Not so many polytopes!

Goodman & Pollack 1986
There are asymptotically far fewer polytopes than we thought

<u>Thm</u>: [Alon 1986]
The number of d-polytopes with n vertices is

$$\leq (n!)^{d^2 + o(1)} = 2^{O(n \log n)}$$

Oriented matroids

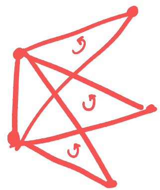
$A = (a_1, \ldots, a_n) \in \mathbb{R}^{d \times n}$ point configuration

chirotope:

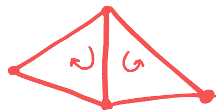
$$\chi : \binom{n}{d+1} \longrightarrow \{+, -, 0\}$$

$$i_1, \ldots, i_{d+1} \longmapsto \operatorname{sign}\left(\det\begin{pmatrix} | & | & & | \\ a_{i_1} & a_{i_2} & \cdots & a_{i_{d+1}} \\ | & | & & | \\ 1 & 1 & & 1 \end{pmatrix}\right)$$

chirotope $\Longrightarrow$ face lattice



vs



$$\boxed{\#\text{ polytopes} \leq \#\text{ chirotopes}}$$

Thm: [Milnor 1964; Thom 1965]

$p_1, \dots, p_n$ d-variable real polynomials of degree $\le D$

The number of sign patterns of $p_1, \dots, p_n$ ranging through $x \in \mathbb{R}^d$ is

$$\le \left(\frac{50 D n}{d}\right)^d$$

$n = \binom{n}{d+1}$

$D = dn \implies$ # polytopes $\le \left(\frac{50\, d \binom{n}{d+1}}{dn}\right)^{dn} = (n!)^{d^2+o(1)} = 2^{O(n \lg n)}$

$d = d$

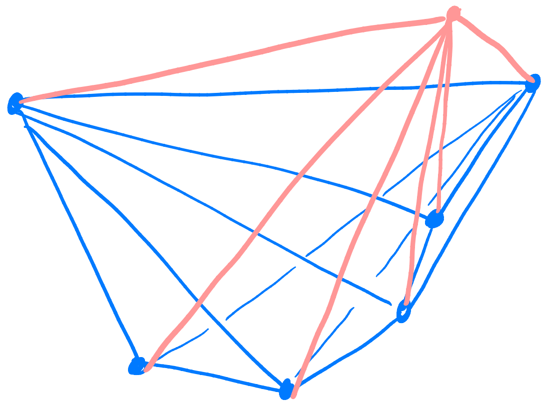
$$\boxed{\text{Most spheres are not polytopal!}}$$

Many polytopes

Thm: [Shemer 1982]

The number of neighborly d-polytopes with n vertices is

$$\geq (n!)^{\frac{1}{2}+o(1)} = 2^{\Omega(n \lg(n))}$$

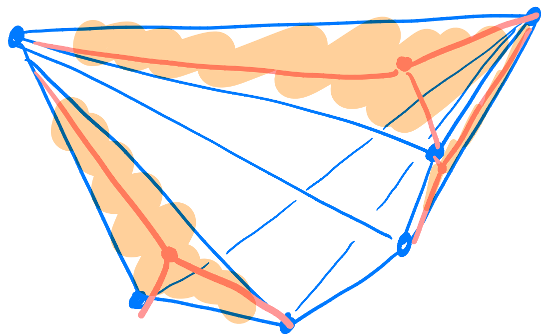


sewing construction

Thm: [Alon 1986]

The number of simplicial d-polytopes with n vertices is

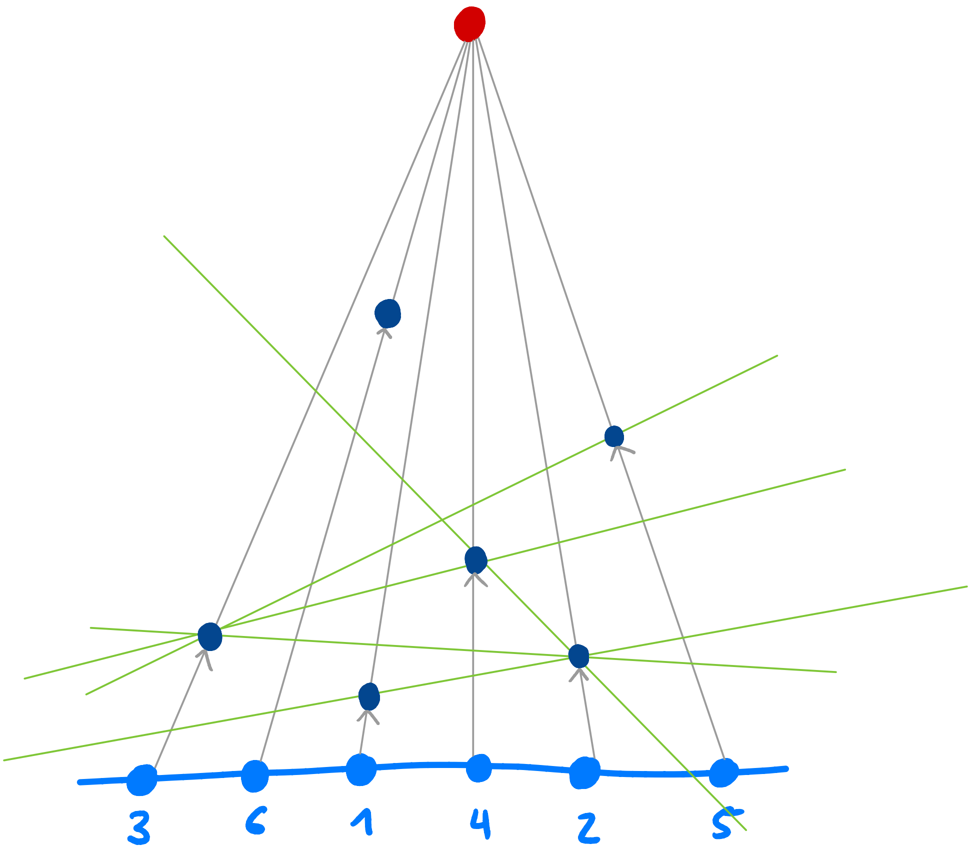
$$\geq (n!)^{\lfloor \frac{d}{4} \rfloor + o(1)}$$



Thm : [P. 2013]

The number of neighborly d-polytopes with n vertices is

$$\geq (n!)^{\lfloor \frac{d}{2} \rfloor + o(1)}$$



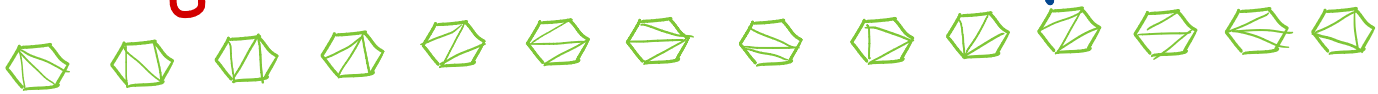
- Positive lexicographic liftings
- Change the order every two dimensions

Thm: [P. - Philippe - Santos 2022+]
The number of simplicial d-polytopes with n vertices is

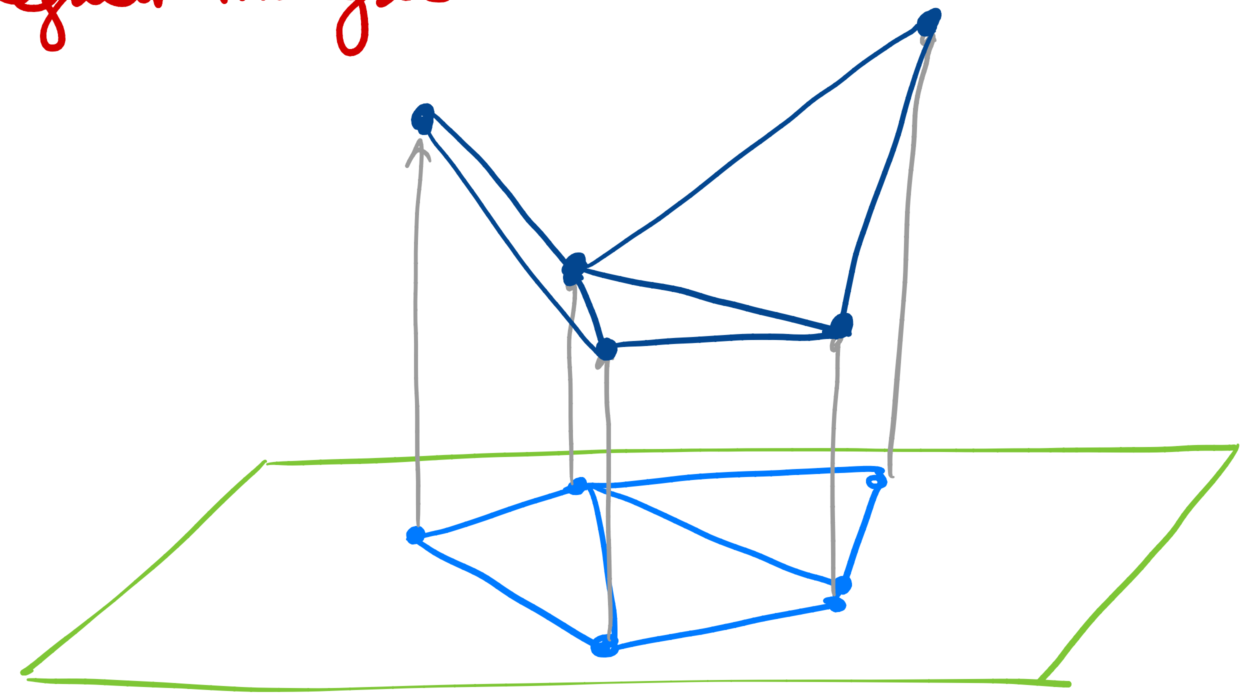
$$\geq (n!)^{d-2+o(1)}$$

By showing that the previous polytopes have many regular triangulations.

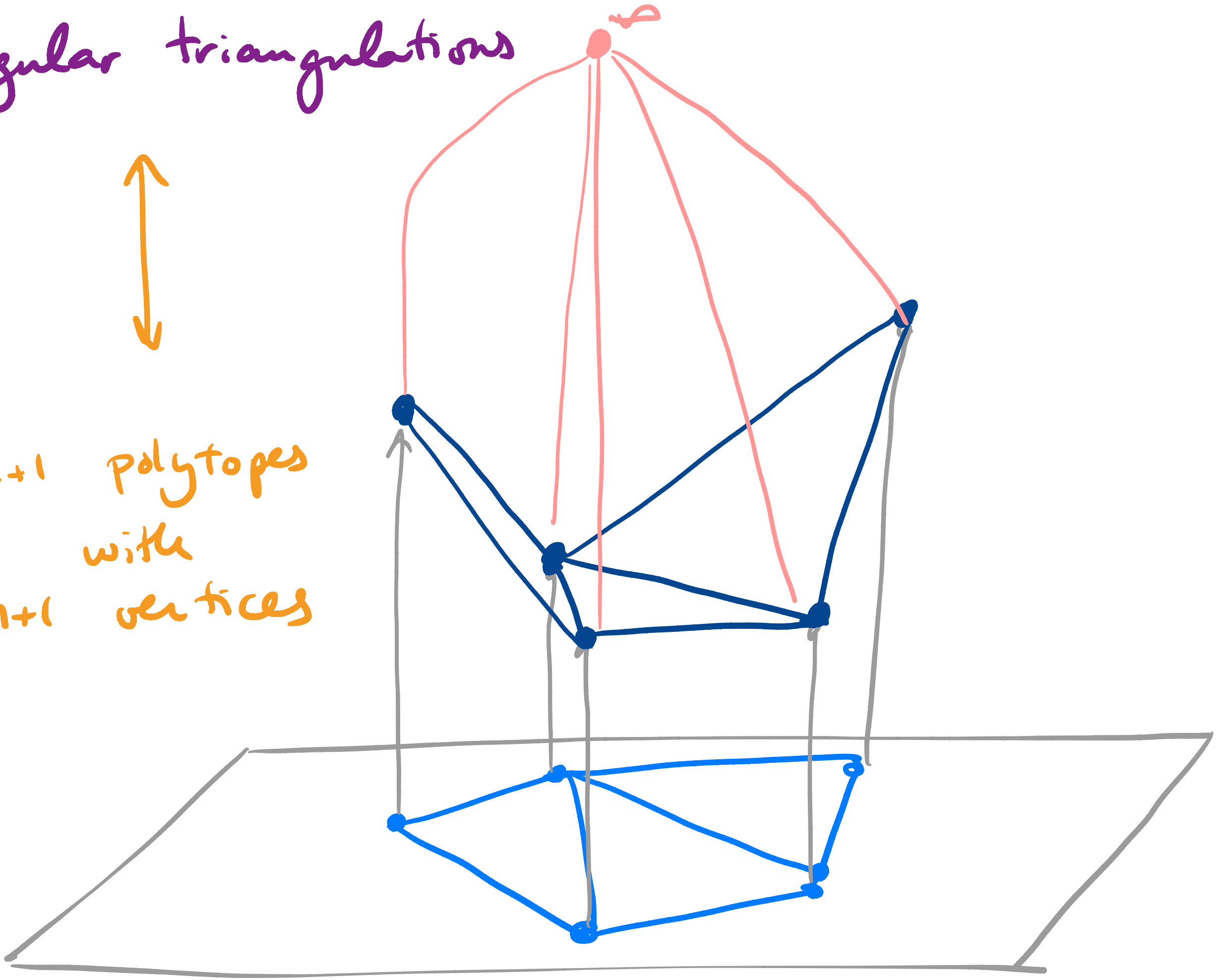
Triangulation: subdivision into simplices



Regular triangulation:

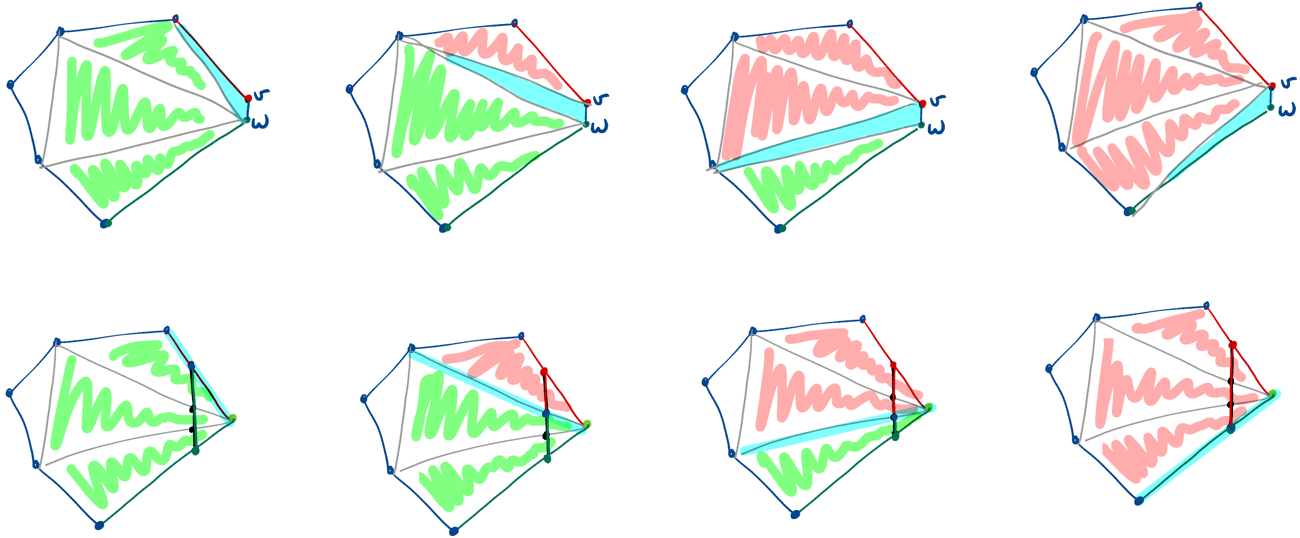


Regular triangulations

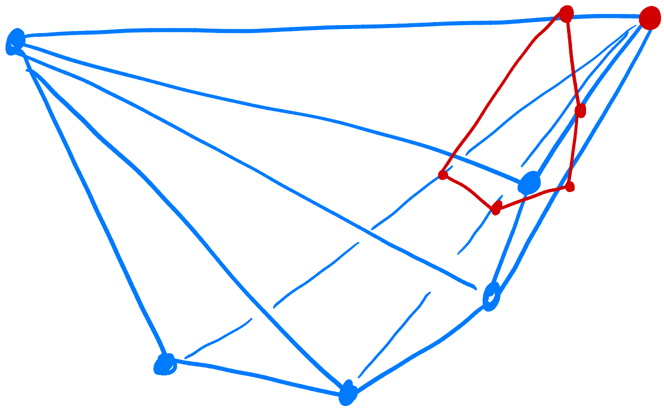


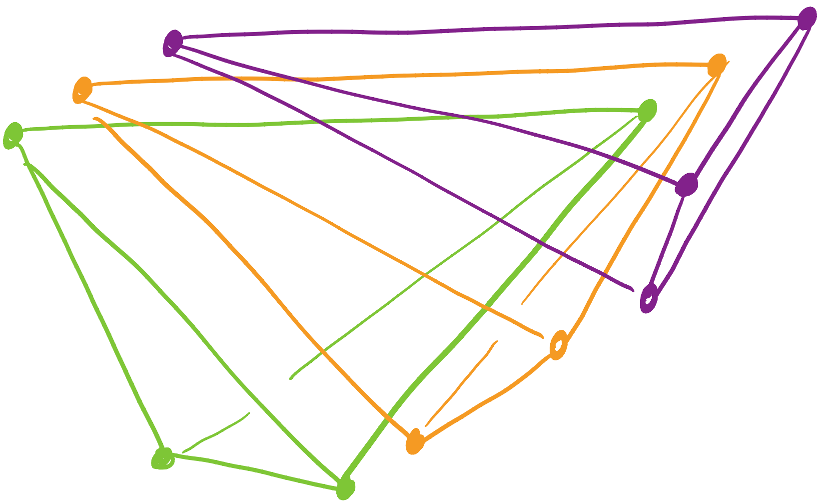
d+1 polytopes
with
n+1 vertices

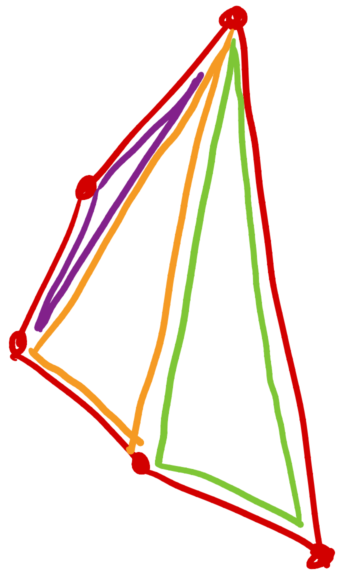
When two vertices v & w are very close,

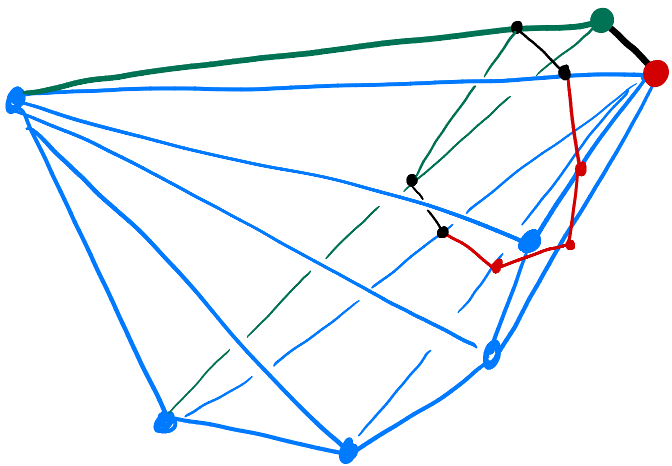


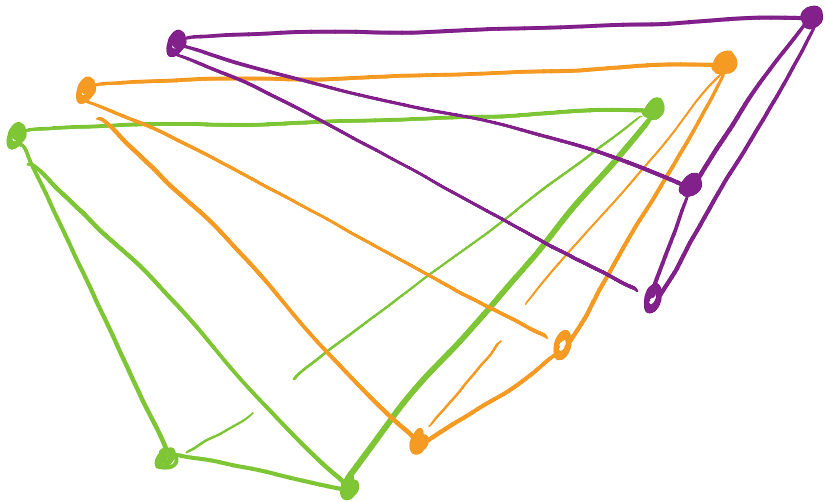
each triangulation of P is determined by a triangulation T of $P \setminus w$ & a section of T/v, its link at v

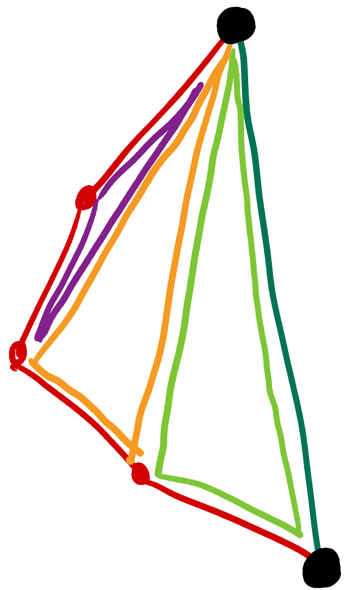


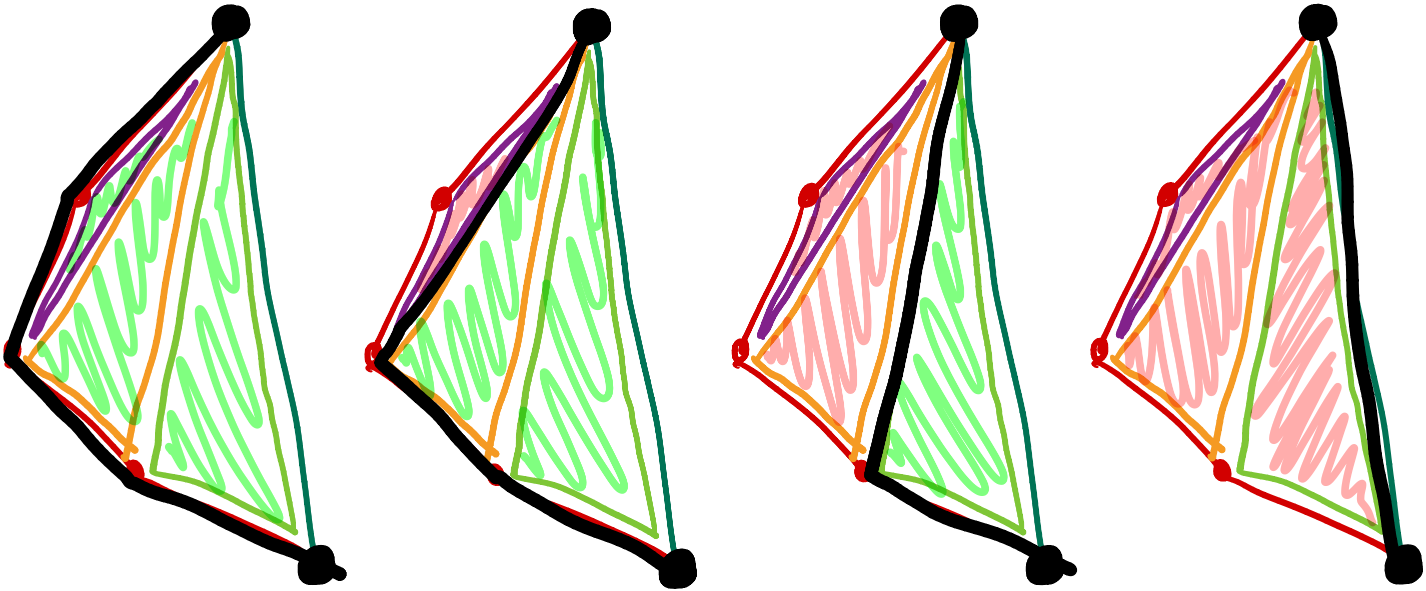


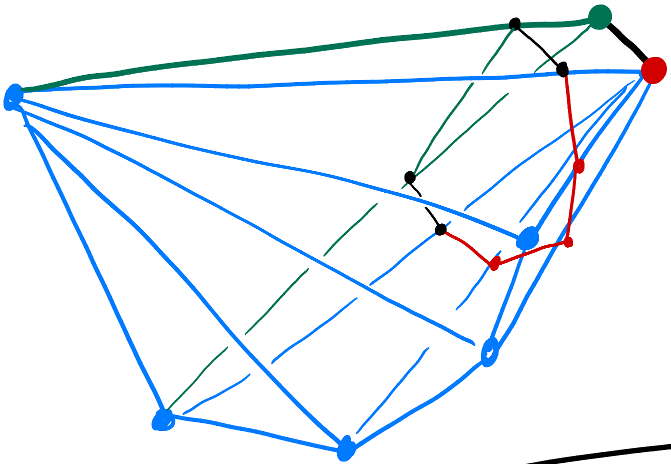


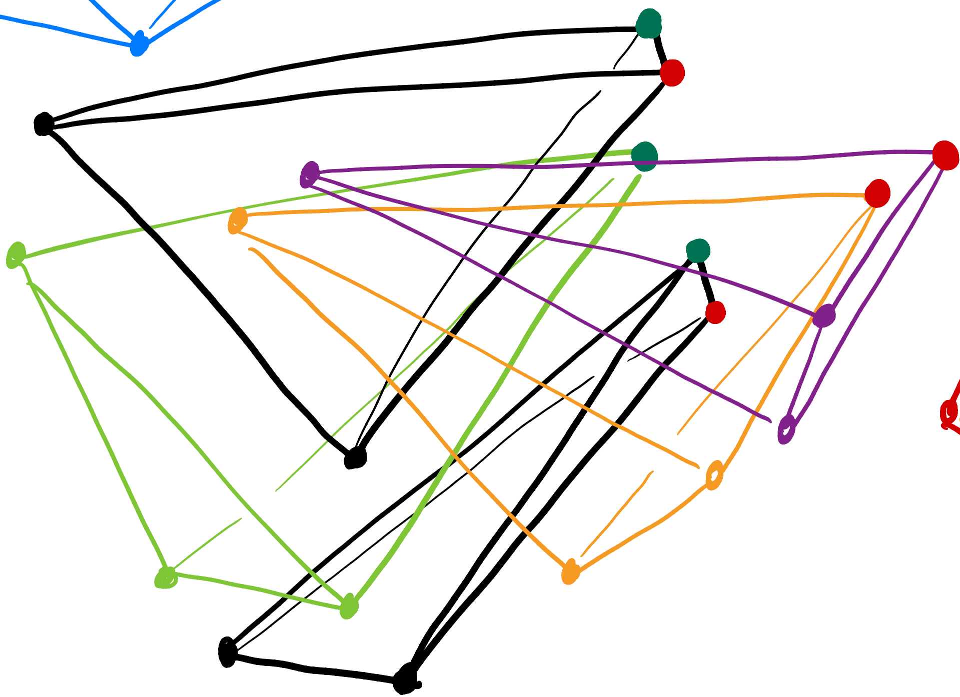


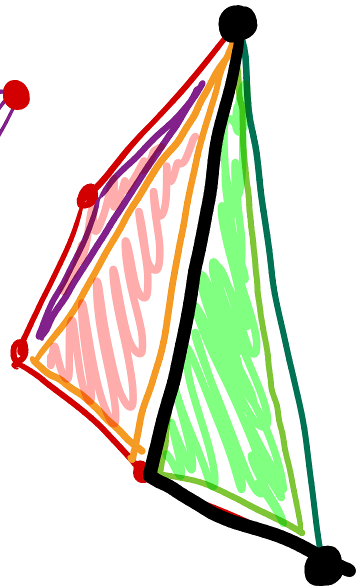


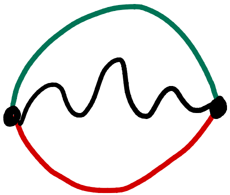




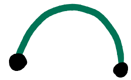


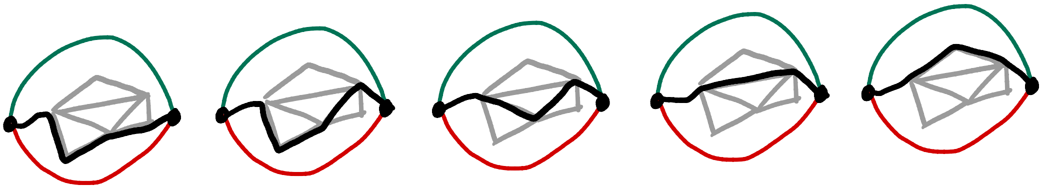






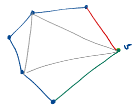
Not every section gives rise to a regular triangulation

There is a path of regular sections from  to  flipping over cells of T/σ

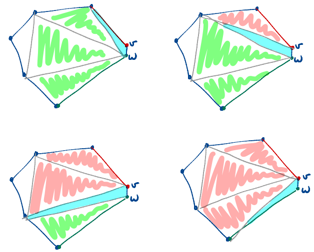


$\Rightarrow$ the number of regular sections is $\geq$ the number of cells

If P/v is neighborly $\Rightarrow$ T/v has many cells



$\Rightarrow$ Each triangulation of $P \setminus w$ extends to many triangulations of P



Use this argument inductively:

$\Rightarrow$ P has many regular triangulations.

$\square$

Summing up:

[GP 1986]
[Alon 1986]

$$n!^{\frac{1}{2}} \le n!^{\lfloor \frac{d}{4} \rfloor} \le n!^{\lfloor \frac{d}{2} \rfloor} \le n!^{d-2} \le \#\,\text{d-polytopes with } n \text{ vertices} \le n!^{d^2}$$

[P.-Philippe-Santos 2022+]

[P. 2013]

[Alon 1986]

[Shemer 1982]

neighborly

(+ o(1) in the exponents)

Some open questions:

- Are there $(n!)^{d^2+o(1)}$ many polytopes?
- How many d-polytopes with n vertices & m facets are there?
- Do most d-polytopes have $O(n^{\lfloor d/2 \rfloor})$ facets?

Thank you!