2-LC triangulated manifolds are exponentially many

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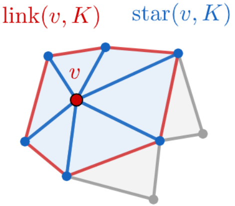
Background picture

- *Facets*: inclusion-maximal faces of a complex.
- *Pure complex*: all facets of the same dimension.
- *Star* of a face σ: the smallest subcomplex containing all facets that contain σ.



- $\mathrm{link}(\sigma, K) := \{\tau \in \mathrm{star}(\sigma, K) : \tau \cap \sigma = \emptyset\}$
- *Triangulation of a smooth d-manifold M*: a *d*-dim simplicial complex whose underlying space is homeomorphic to *M*.
- *d-sphere*: a triangulation of the *d*-dimensional sphere.
- *d-pseudomanifold*: a *d*-dim pure simplicial regular CW-complex where each $(d-1)$-cell is in ≤ 2 facets.

Big picture

Gromov's question (2000)

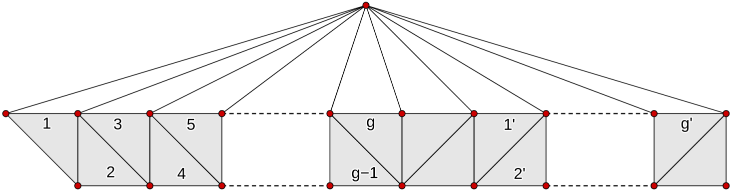
How many triangulations of the 3-sphere with N tetrahedra are there?

- Two triangulations are equivalent $\iff$ same face poset.
- Exponentially many?
- Crucial for discrete version of quantum gravity
  - If yes, all good
  - If no, divergence issues

Too big picture

Theorem (Folklore)

There are more than exponentially many surfaces with N triangles.

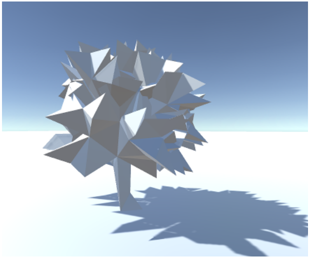


Corollary (via coning)

There are more than exponentially many 3-pseudomanifolds with N tetrahedra.

Locally constructible picture

- LC manifolds are those obtainable from a tree of *d*-simplices by recursively gluing two *adjacent* boundary facets.
- Mogami manifolds: ... gluing two *incident* ...
- All shellable spheres are LC.



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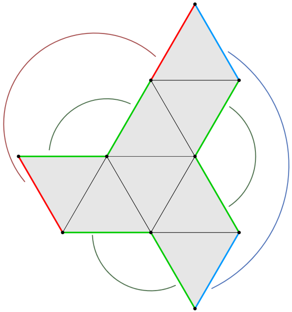

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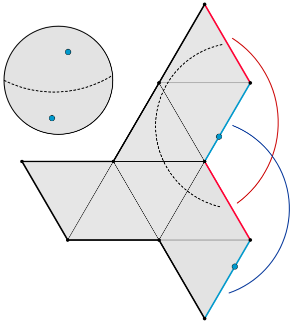


Locally constructible picture

Theorem (Durhuus–Jonsson 1995; Benedetti–Ziegler 2011)

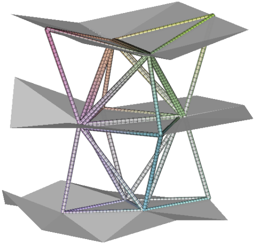
LC triangulations of d-manifolds with N facets are at most $2^{d^2 N}$.

- Works also for LC pseudomanifolds.

Theorem (Mogami 1995)

Mogami triangulations of 3-manifolds with N facets are exponentially many.

- (d=2) Surfaces with fixed genus (Tutte 1962)
- (d=3) Causal triangulations (Durhuus–Jonsson 2014)
- (any d) Bounded geometry (Adiprasito–Benedetti 2020)



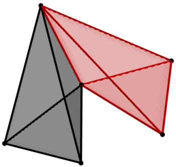
- Triangulations with bounded discrete Morse vector (Benedetti 2012)
  - contains all classes above
  - does *not* contain Mogami triangulations

Turning pictures into a movie

Definition (Benedetti–P. 2022)

t-LC d-manifolds are those obtainable from a tree of d-simplices by recursively gluing two boundary facets whose intersection has dimension at least $d - 1 - t$.

- 1-LC the same as LC
- 1-LC $\subset$ 2-LC $\subset \cdots \subset$ d-LC
- All connected d-manifolds are d-LC



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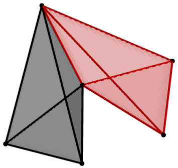

Main Theorem (Benedetti–P. 2022)

2-LC triangulations of d-manifolds with N facets are at most $2^{\frac{d^3}{2}N}$.

Special effects – Coning

Theorem (Benedetti–P. 2022)

Cones over t-LC d-pseudomanifolds are t-LC.

$\Rightarrow$ 2-LC d-pseudomanifolds more than exponentially many!

- Unlike the Benedetti-Ziegler result, our result really uses the *manifold* assumption: without it, it's false.

Crucial facts for our proof

- Links of $(d-3)$-faces in a manifold are homeomorphic to S^2 or a disk.
- Planar gluings lead to count by Catalan numbers.

- Our proof makes precise and extends to all dimensions the intuition for $d = 3$ by Mogami.

- A d-dimensional complex C is called *[homotopy-]Cohen–Macaulay* if for any face F, for all $i < \dim \operatorname{link}(F, C)$, $[\pi_i(\operatorname{link}(F, C)) = 0]$ $H_i(\operatorname{link}(F, C)) = 0$.
- *Constructible simplicial complex* is defined inductively:
  - every simplex, and every 0-complex, is constructible;
  - a d-dim pure simplicial complex C that is not a simplex is constructible if and only if it can be written as $C = C_1 \cup C_2$, where C_1 and C_2 are constructible d-complexes, and $C_1 \cap C_2$ is a constructible $(d-1)$-complex.

Another famous picture

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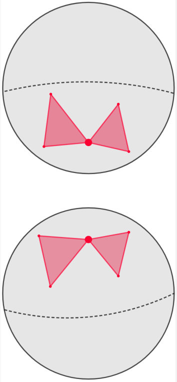
Results to generalize

- Constructible manifolds are LC. (Benedetti–Ziegler 2011)
- Constructible complexes are homotopy-Cohen–Macaulay. (Hochster 1972)

Definition (Benedetti–P. 2022)

Let $0 < t \leq d$ be integers. *t-constructible* d-dimensional simplicial complexes defined inductively:

- every simplex is t-constructible;
- a 1-dimensional complex is t-constructible if connected;
- a d-dimensional pure simplicial complex C that is not a simplex is t-constructible if $C = C_1 \cup C_2$, where C_1 and C_2 are t-constructible d-complexes, and $C_1 \cap C_2$ is a $(d-1)$-complex whose $(d-t)$-skeleton is constructible.



Final picture

Theorem (Benedetti–P. 2022)

t-constructible pseudomanifolds are t-LC.

Theorem (Benedetti–P. 2022)

The $(d - t + 1)$-skeleton of a t-constructible d-complex is homotopy-Cohen–Macaulay.

(In other words, t-constructible d-complexes have (homotopic) depth $> d - t$.)