Enumerating all Triangulations up to Symmetry

Or: The Power of Order Rightly Used

Jörg Rambau

UNIVERSITÄT BAYREUTH

Lehrstuhl für Wirtschaftsmathematik

September 5–9, 2022
Workshop “Geometry meets Combinatorics”
Bielefeld

Agenda

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Given

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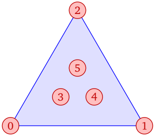
Given A point configuration



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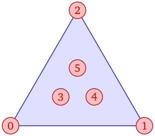

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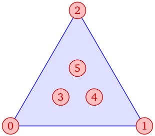

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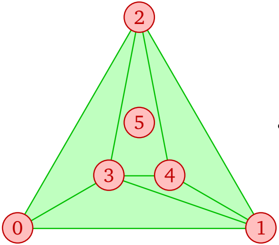

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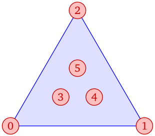

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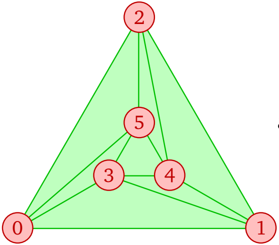

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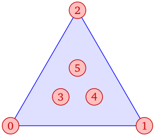

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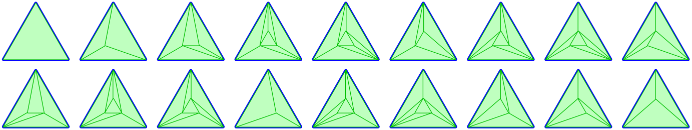

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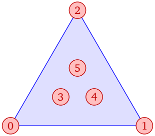

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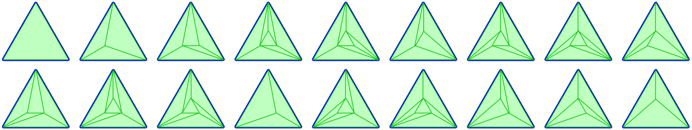

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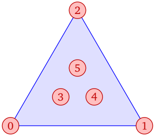

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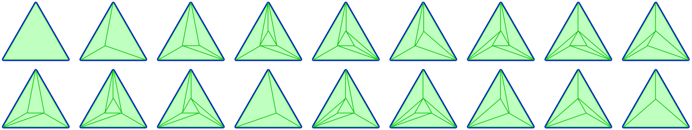

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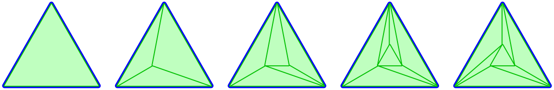

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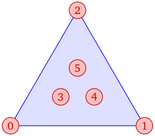

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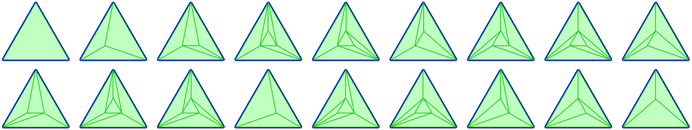

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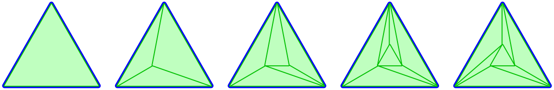

This Talk Enumerate them with a computer.

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New: R. 2022	Parallel symmetric lexicographic subset reverse search for all triang's (new C++-code TOPCOM 1.x.x)

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- return counter.

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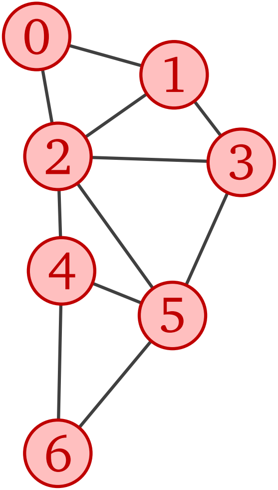

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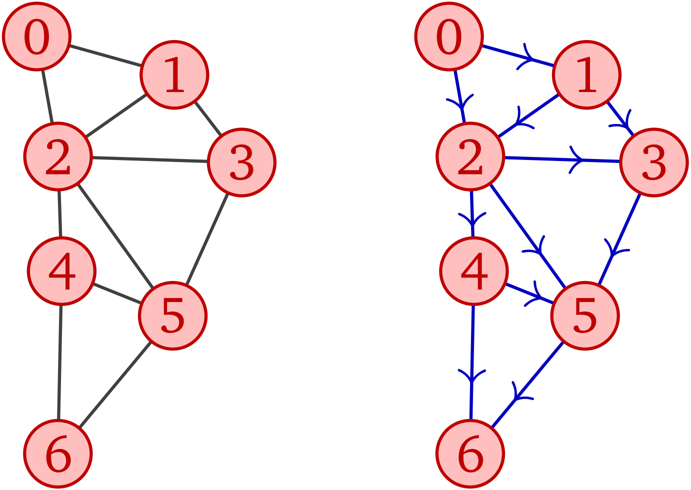

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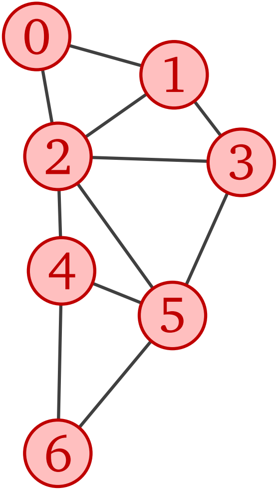

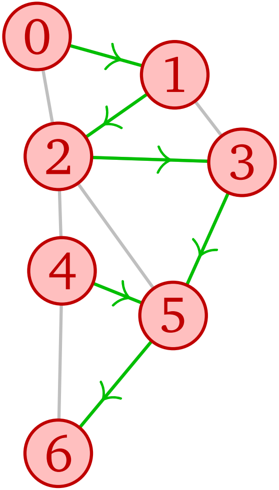

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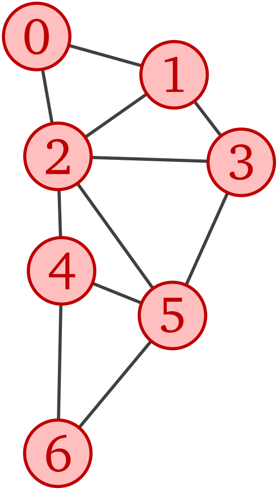

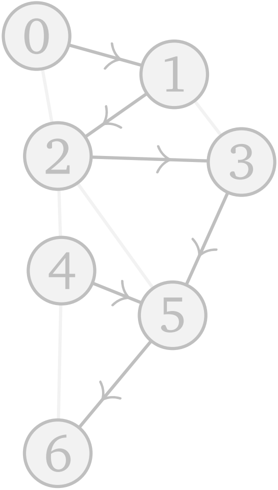

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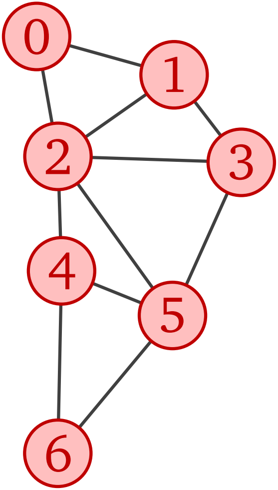

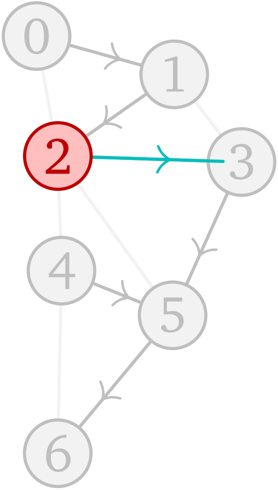

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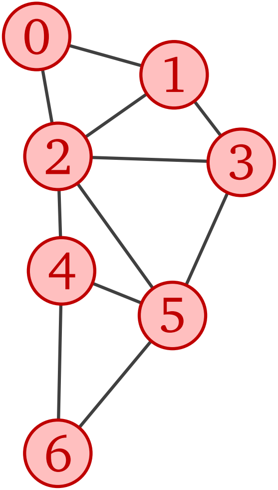

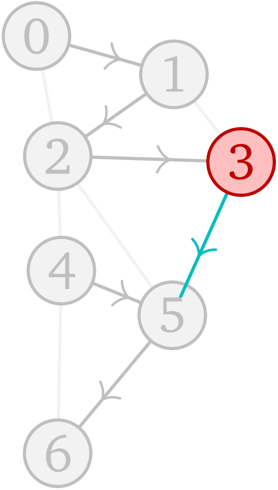

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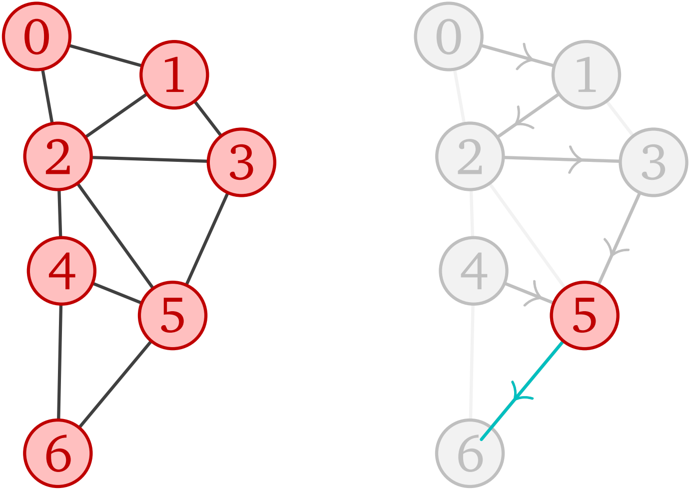

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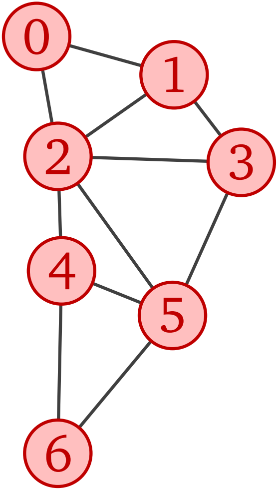

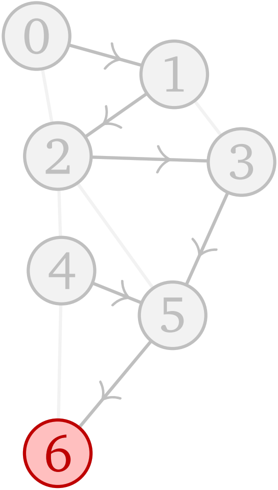

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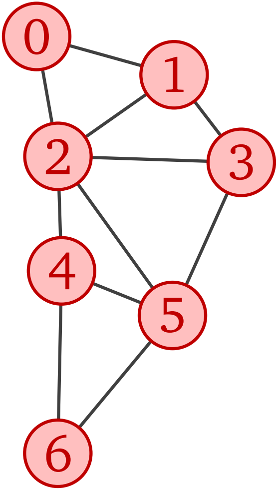

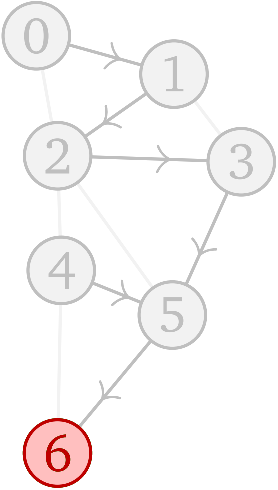

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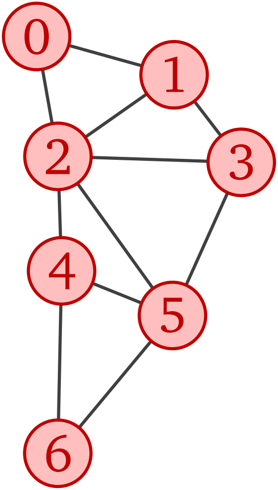

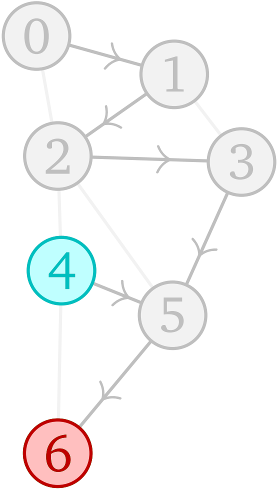

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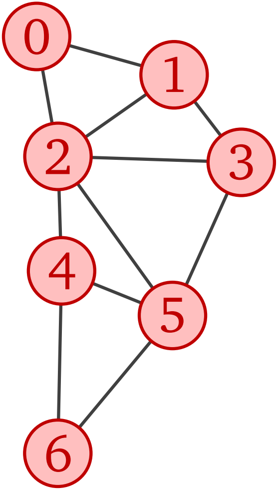

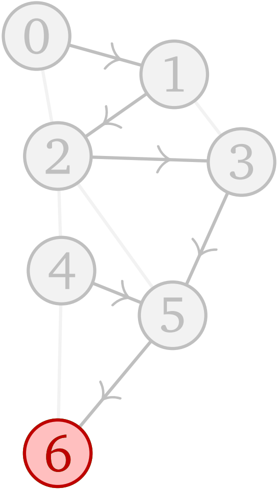

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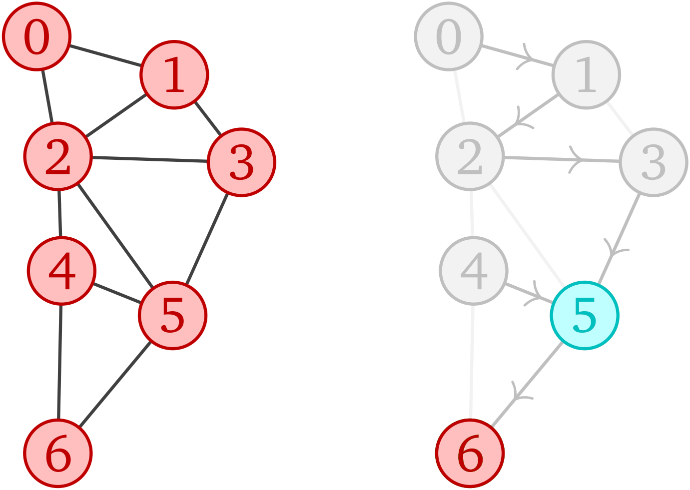

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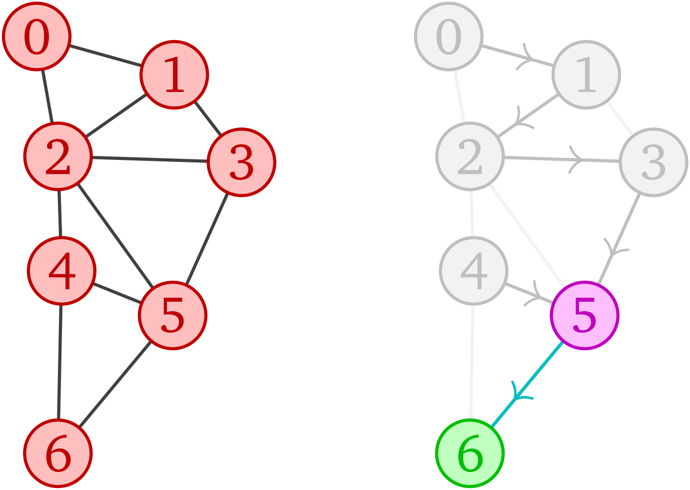

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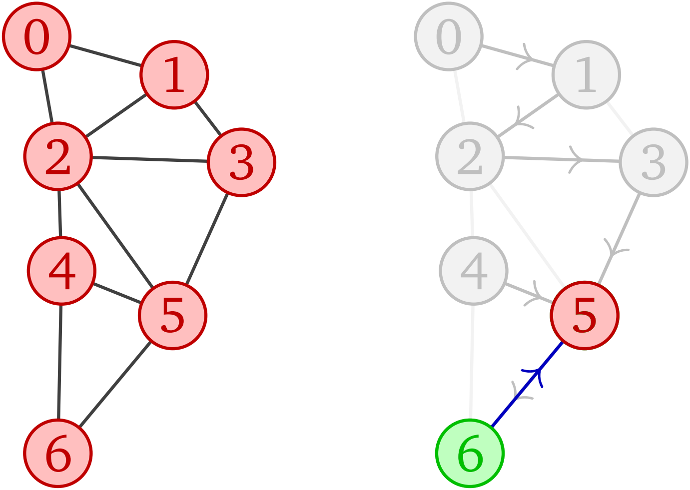

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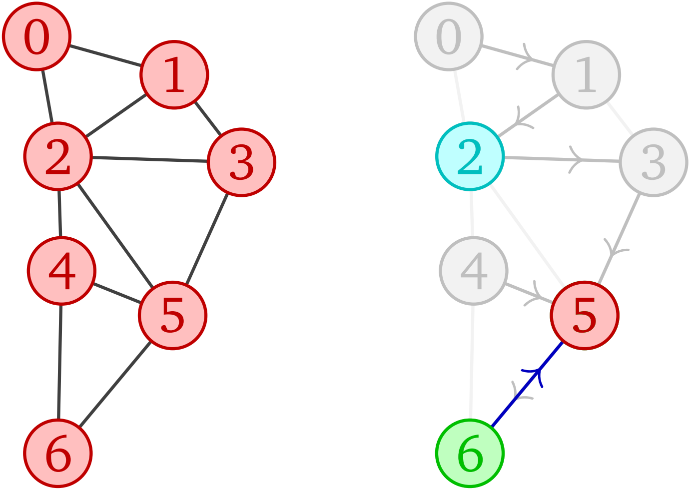

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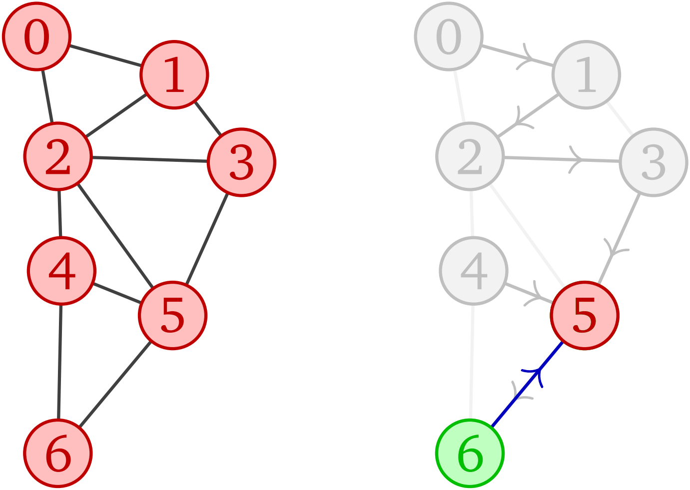

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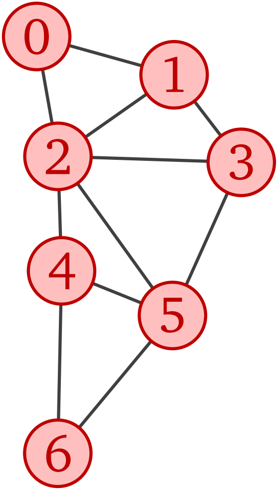

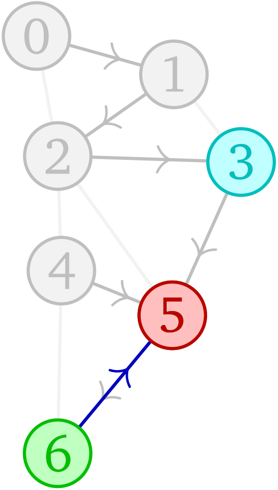

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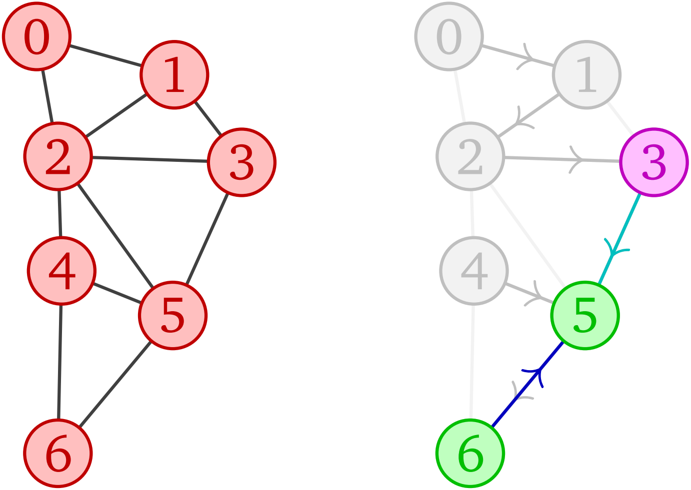

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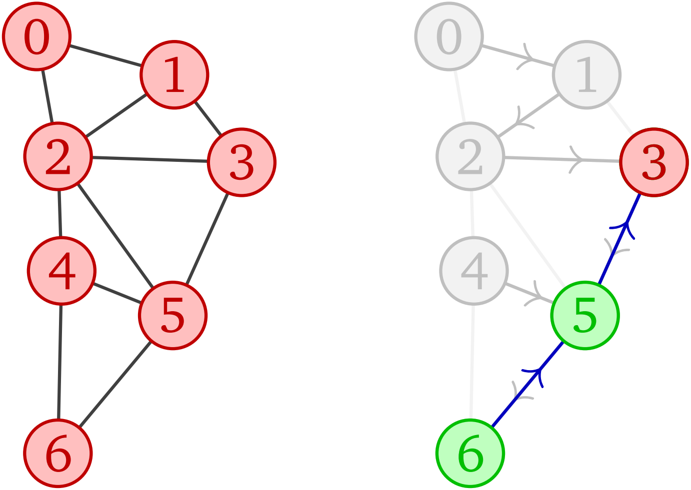

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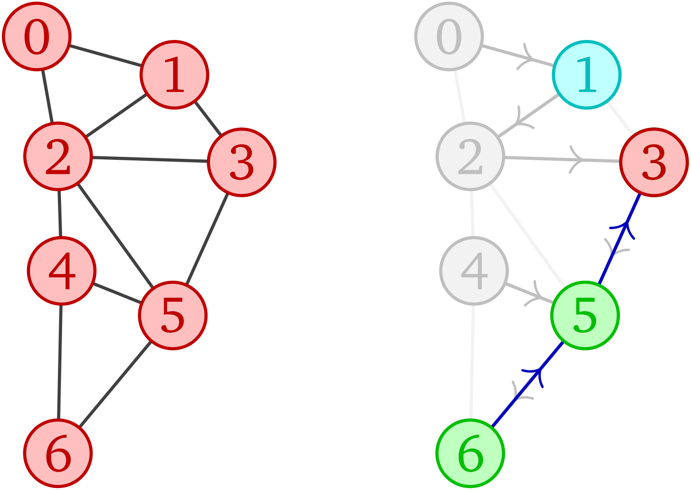

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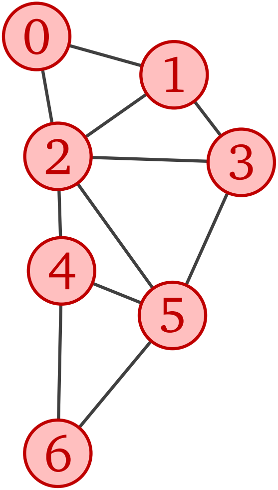

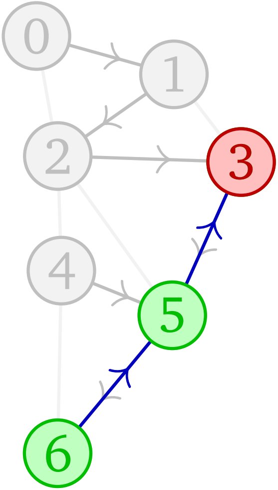

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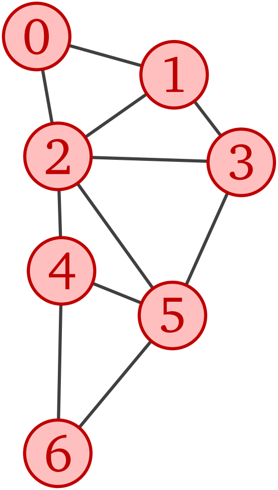

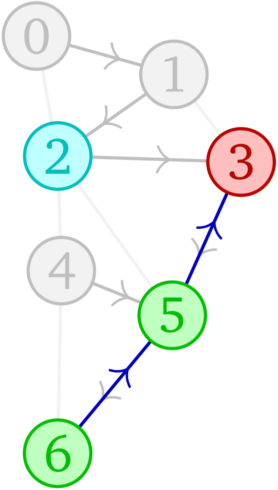

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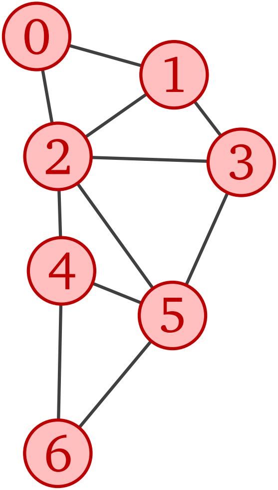

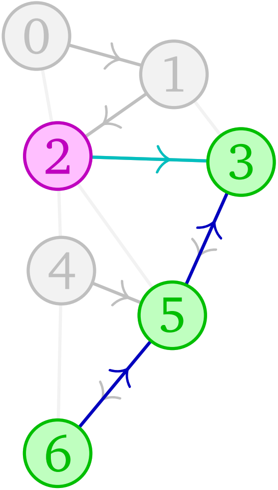

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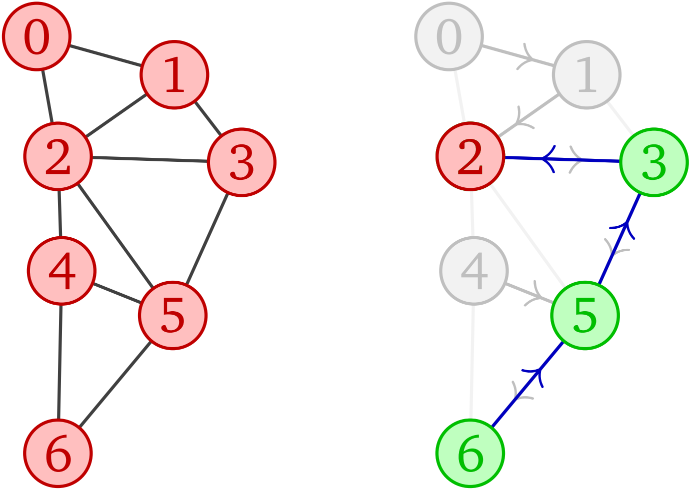

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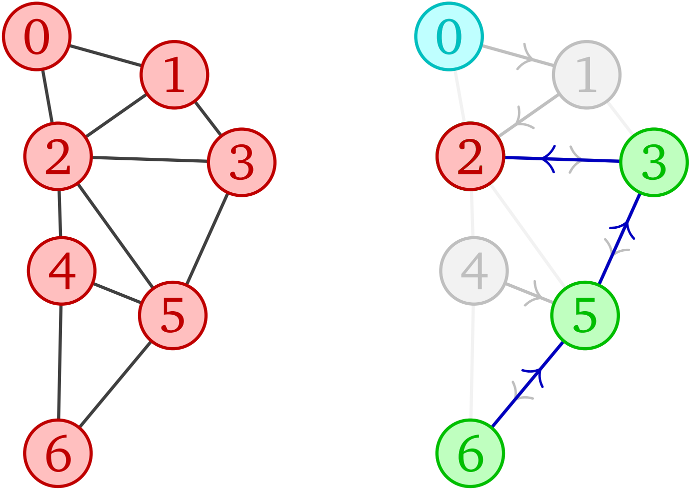

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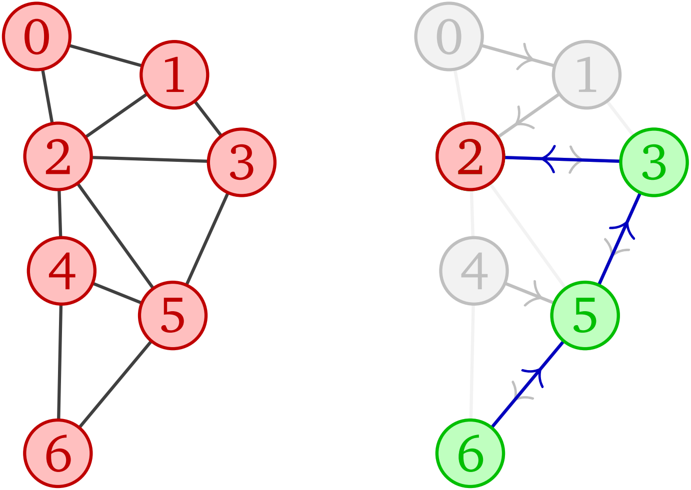

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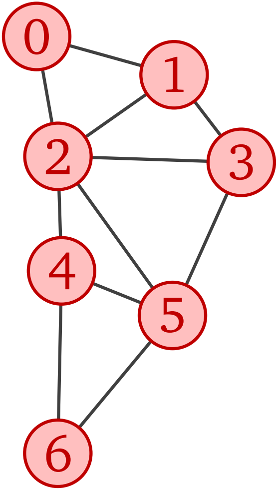

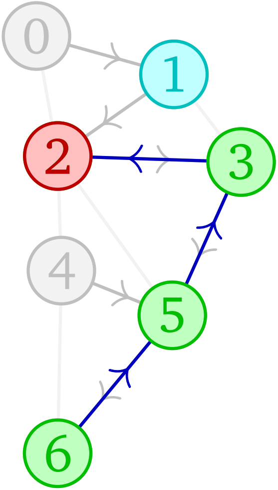

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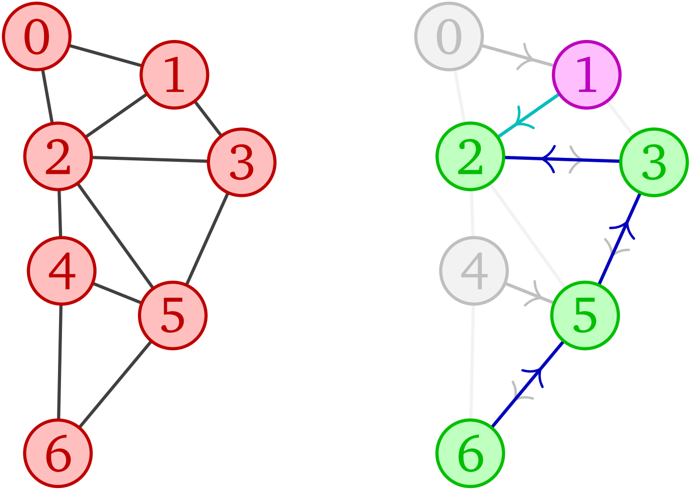

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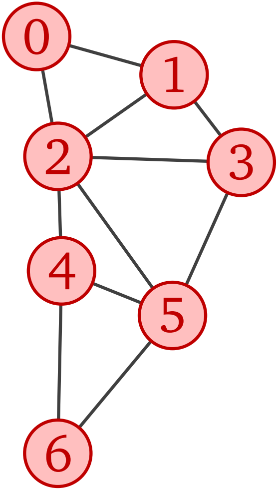

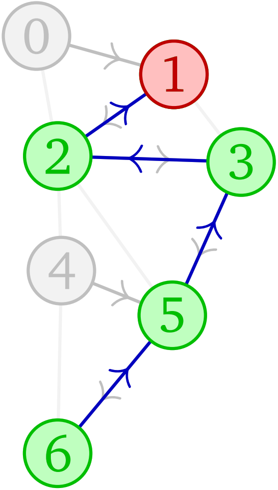

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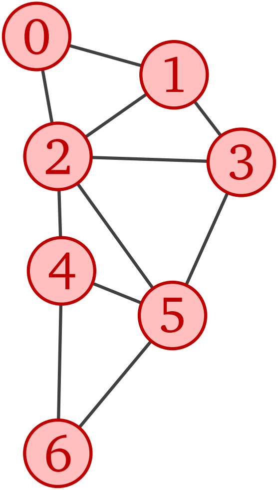

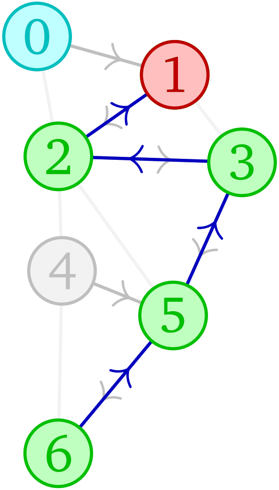

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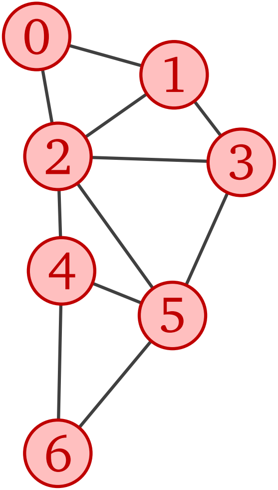

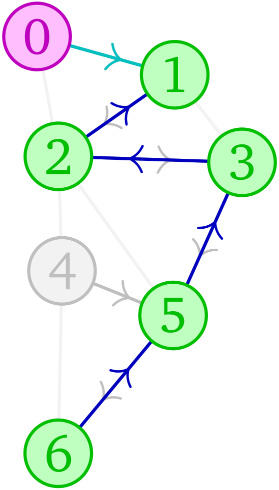

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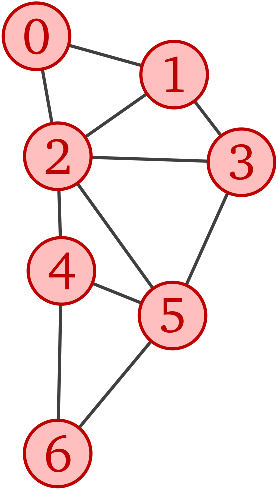

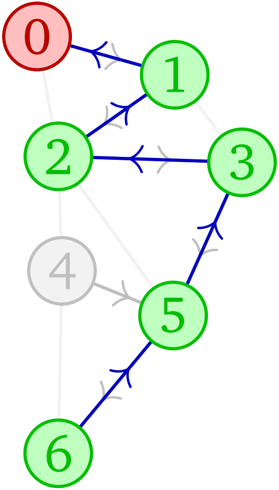

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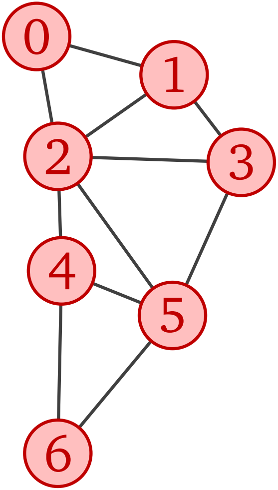

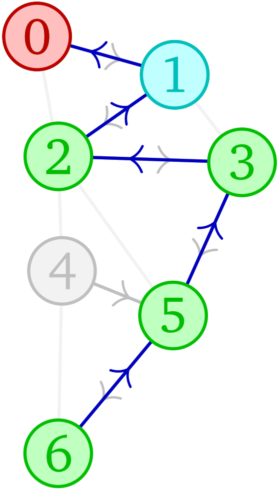

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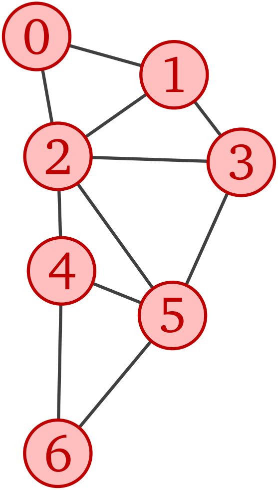

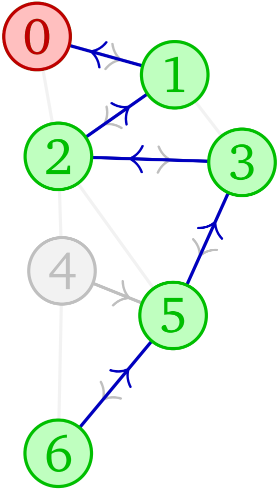

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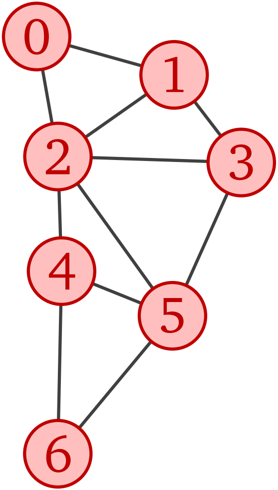

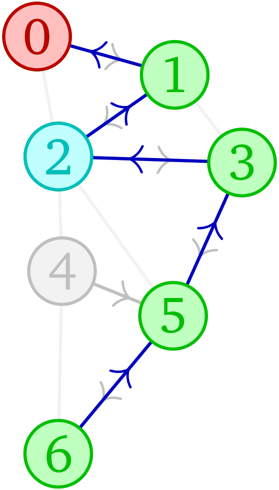

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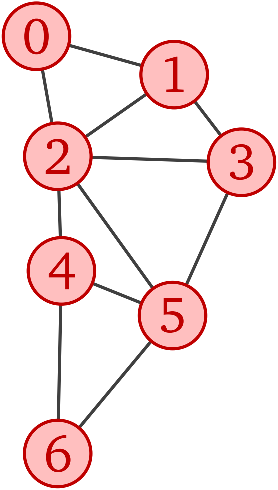

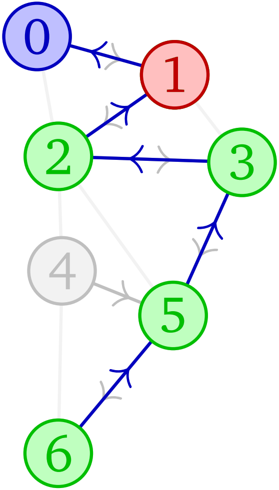

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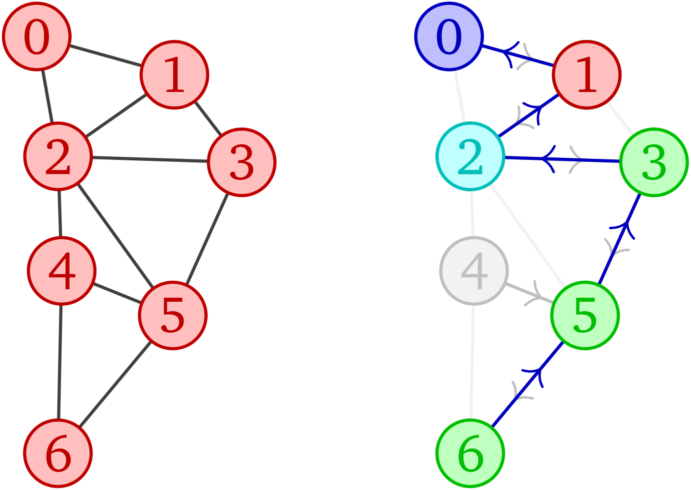

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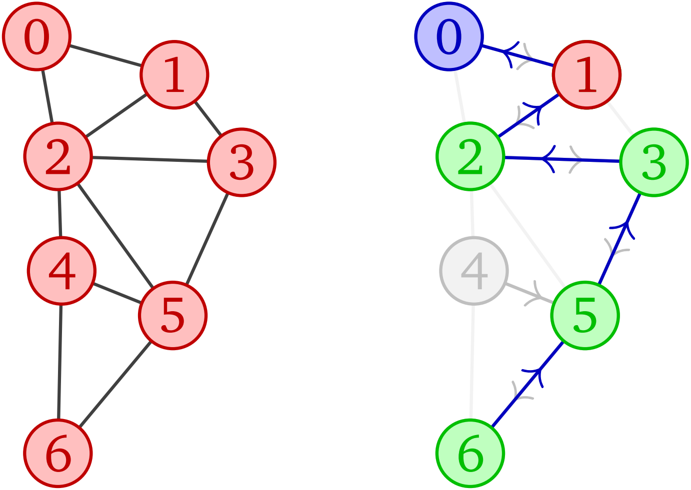

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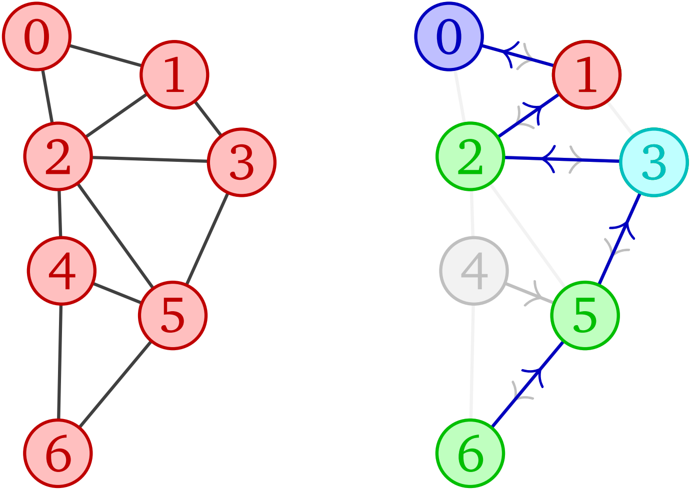

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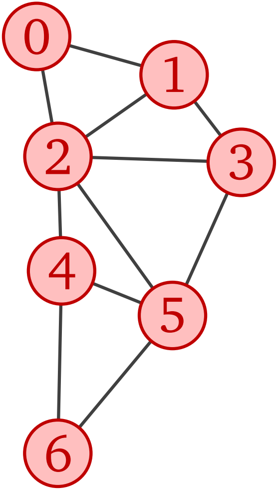

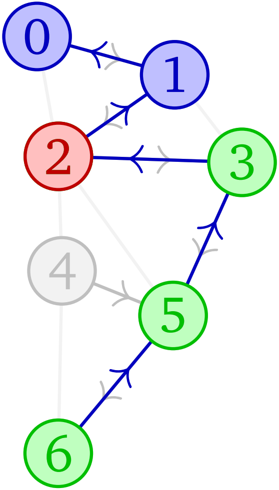

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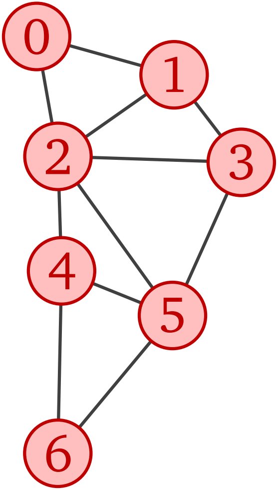

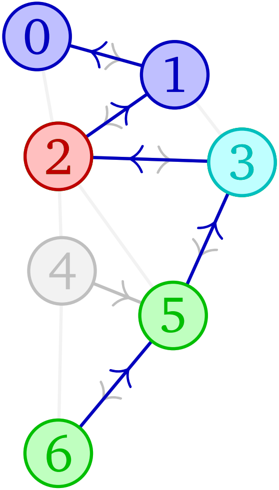

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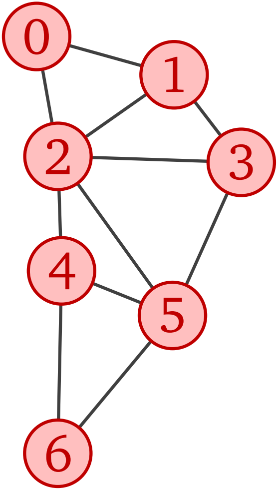

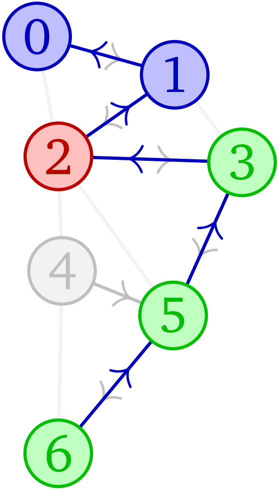

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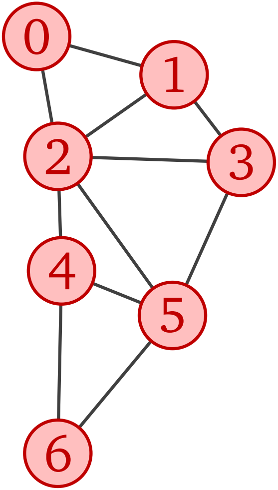

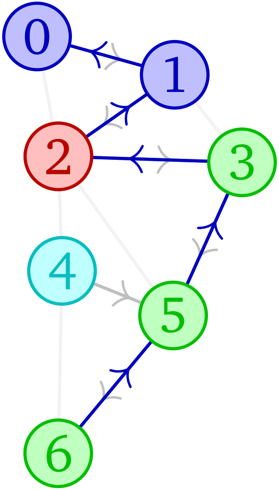

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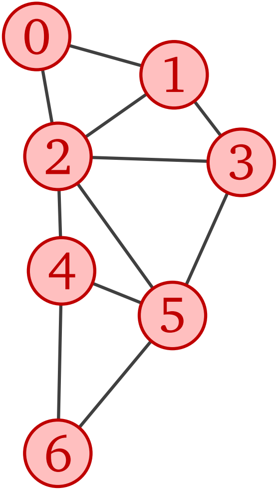

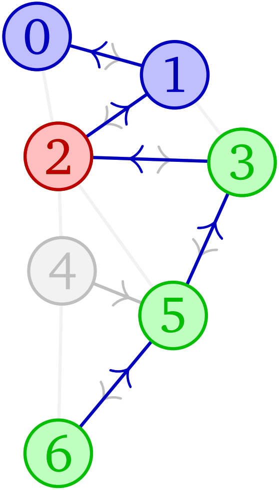

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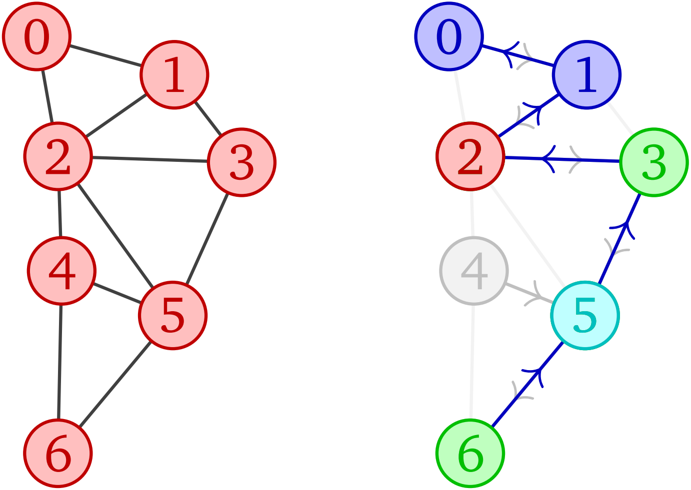

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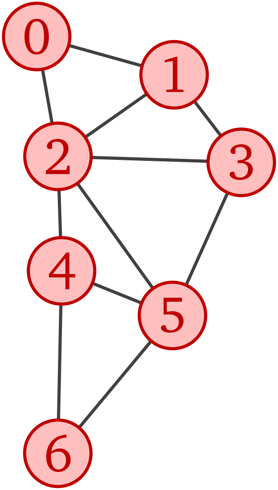

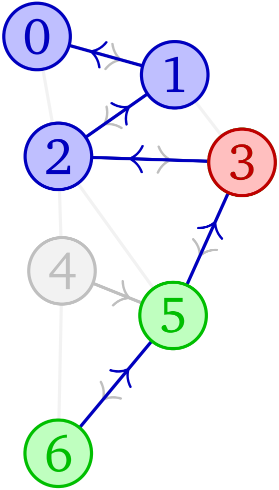

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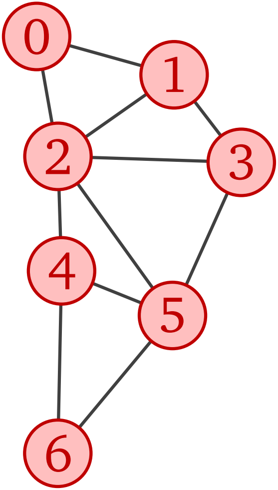

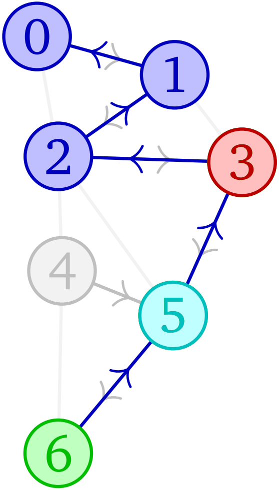

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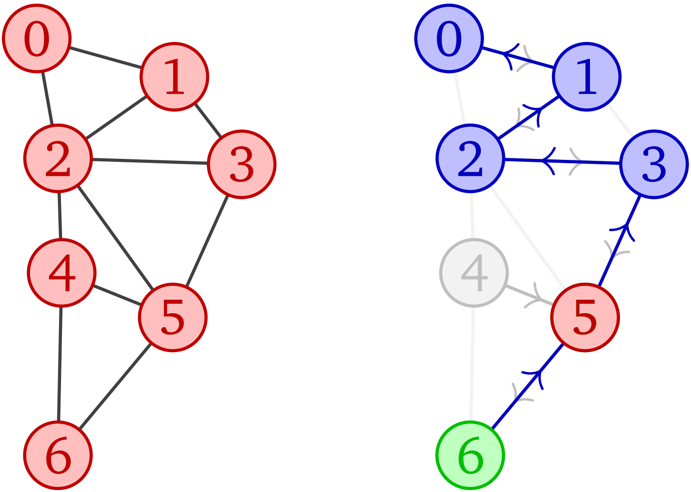

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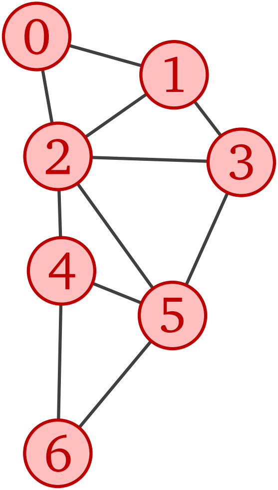

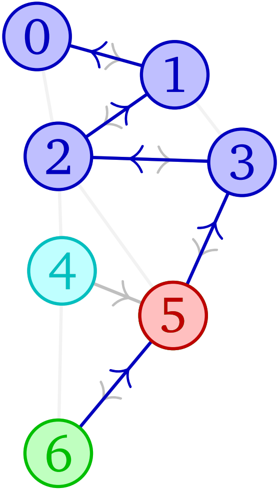

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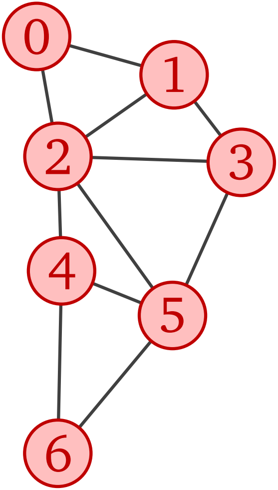

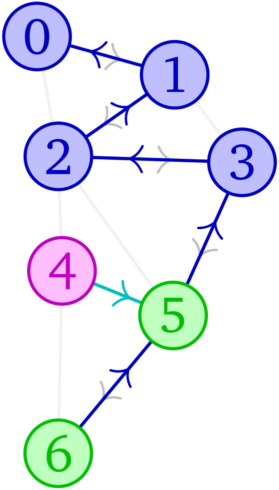

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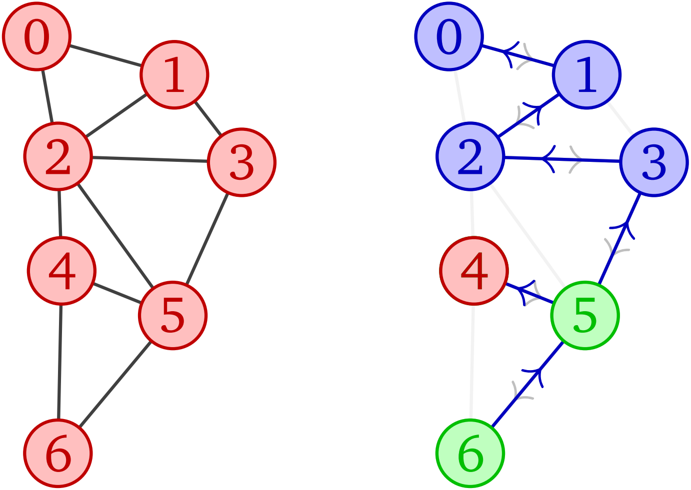

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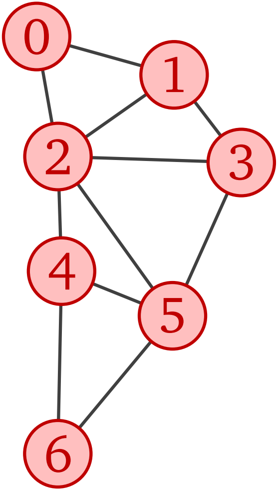

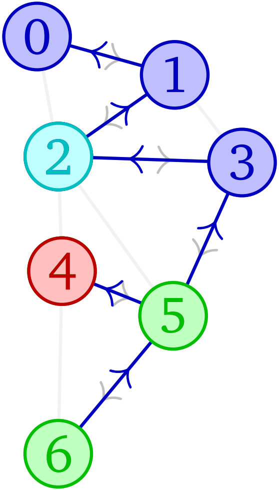

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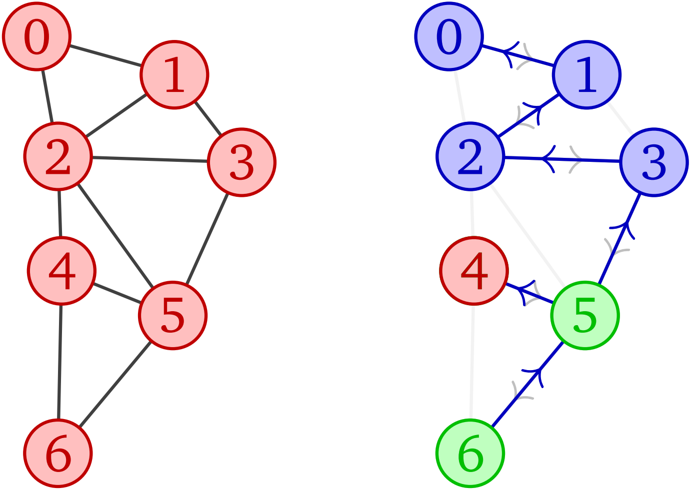

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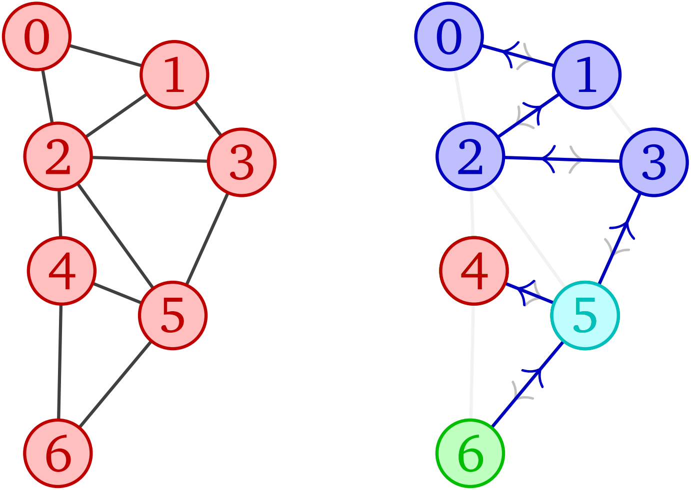

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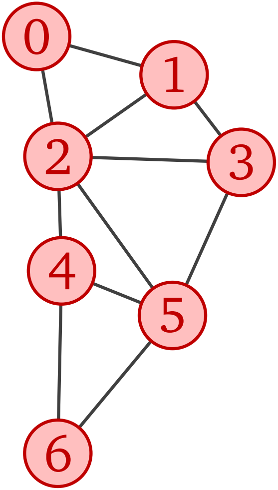

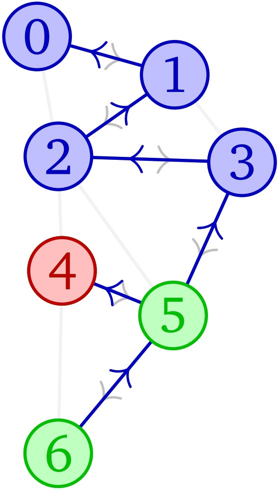

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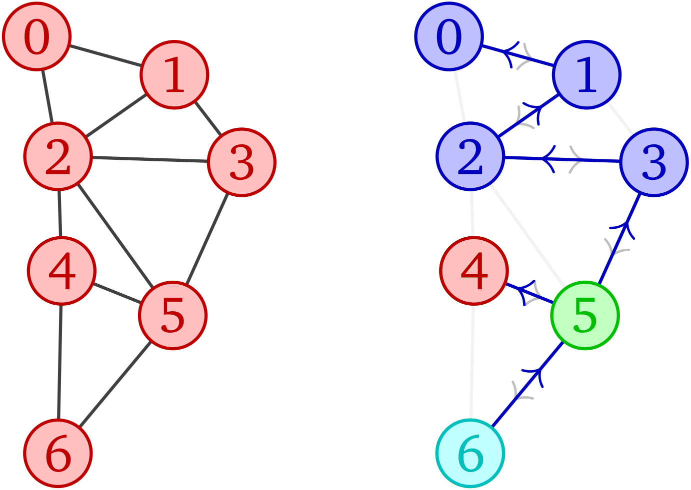

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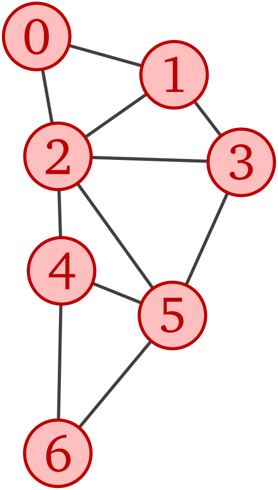

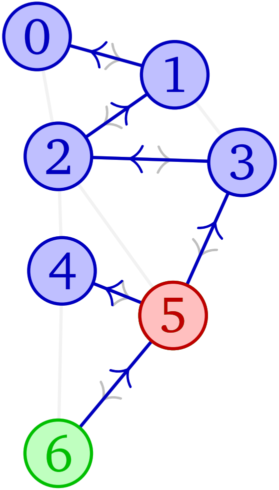

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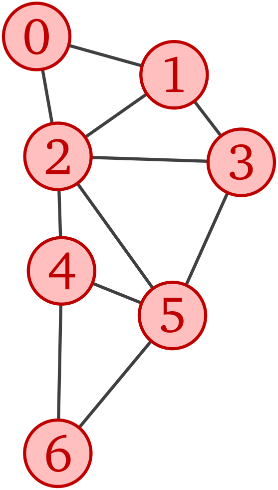

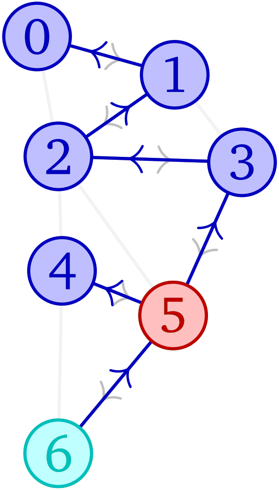

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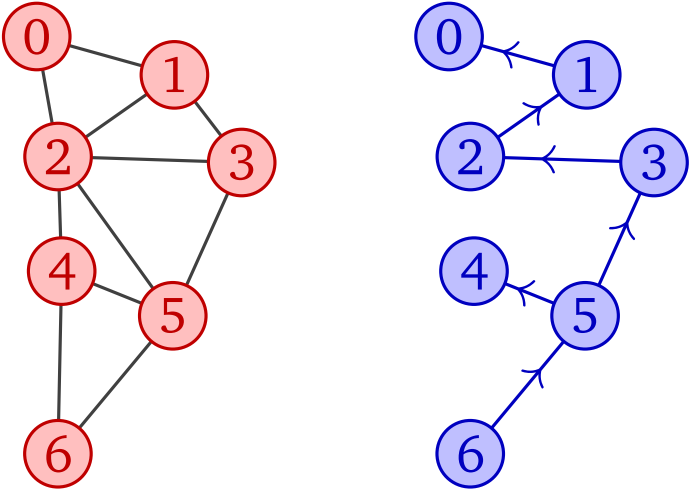

Reverse Search on Orbits

Canonical Representatives

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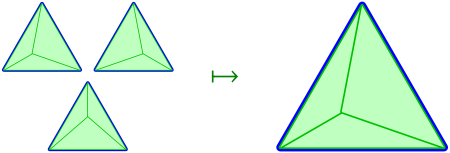
Function "G-Orbits $\longrightarrow$ Elements", e.g.,



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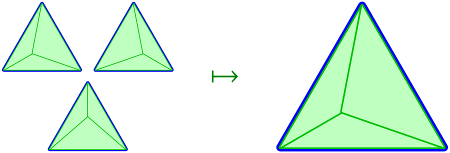
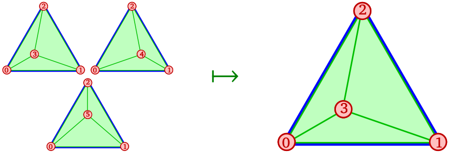


RS-Consistent Choice

orbit $\mapsto$ objective-minimal sink in orbit, e.g.,



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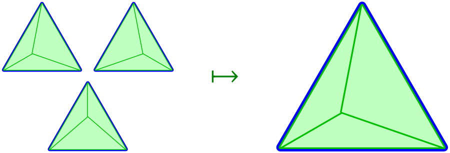

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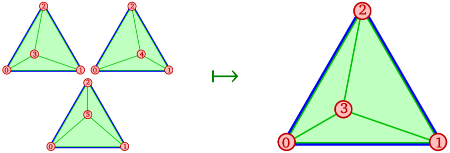

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new pivot := canonical representative $\circ$ old pivot.

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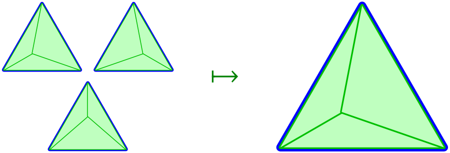

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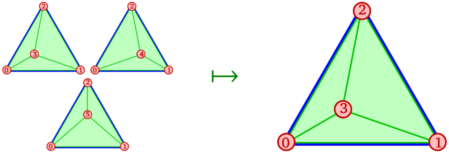

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Can enumerate orbits by RS on orbits [Imai et al. 2002].

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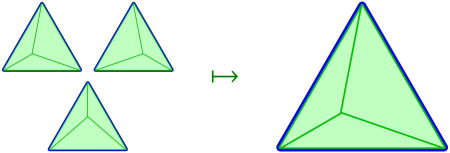

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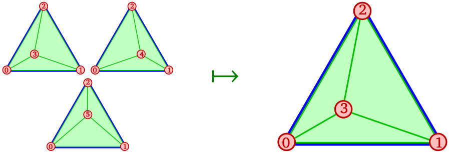

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Bottleneck

Compute canonical representatives.

Agenda

Representation of Objects as Subsets

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Generic Algorithm: Symmetric LSRS

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Subset S lex-min in its orbit
$\implies S \setminus \max S$ lex-min in its orbit.

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$\longrightarrow$ symLSRS($\emptyset$) lex-enumerates all orbit-lex-min objects

Agenda

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$$\text{A symmetry } \pi \text{ lex-decreases a subset } S \iff \mathrm{crit}_S(\pi) \in \pi(S).$$

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Convention

All $s \in \{1, \dots, n_s\}$ and $f \in \{1, \dots, n_f\}$ are called simplices and facets, resp.
All $T \subseteq \{1, \dots, n_s\}$ with pairwise proper intersections are called partial triangulations.

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A partial triangulation T is not lex-extendable

$$\Longleftarrow$$

there is $\{f \in \mathcal{F}(T)\}$ not contained in any $\{s \in \mathcal{A}(T)\}$.

Ingredient II: Lex-pruning/Lex-Breaking

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Gain

One integer comparison instead of many subset tests.

Effectivity

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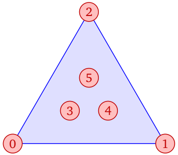
Comparison (with lex-breaking):



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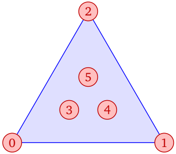
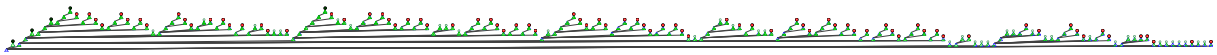


No Pruning



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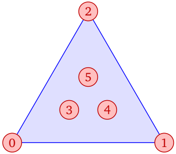

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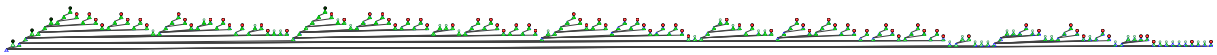
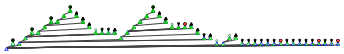


Full Pruning



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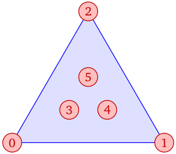

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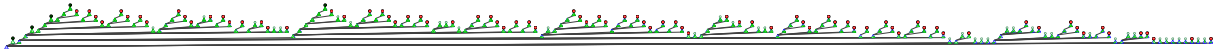

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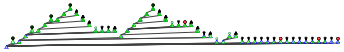
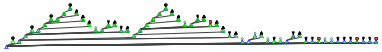


Lex Pruning



Computational Results for Triangulations

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MPTOPCOM
Flip-Based
CPU Times
(16/40 Threads)

[Jordan & Joswig & Kastner 2018]

Point Conf.	# Triang's	# Orbits	CPU time [hh:mm:ss]
$[0, 1]^4$	92,487,256	247,451	00:01:56
$3D_3$ (reg./full/output)	22,201,684,367	925,148,763	96:00:00

Computational Results for Triangulations

MPTOPCOM
Flip-Based
CPU Times
(16/40 Threads)

[Jordan & Joswig & Kastner 2018]

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TOPCOM 1.0.8
Subset-Based
CPU Times
(16 Threads)

[R. 2022]

Point Conf.	# Triang's	# Orbits	CPU time [hh:mm:ss]
$[0,1]^4$	92,487,256	247,451	00:00:04
$3D_3$ (output)	22,201,684,367	925,148,763	01:05:11
$3D_3$ (count)	22,201,684,367	925,148,763	00:21:02
$3D_3$ (regular)	21,861,522,799	910,974,879	20:21:53
$3D_3$ (full)	511,052,427	21,302,400	00:01:01
$3D_3$ (unimod.)	346,903,379	14,459,488	00:00:39
Dodecahedron	1,533,079,037,570	12,775,757,027	11:11:48
Pyritohedron	32,734,029,351,118	1,363,918,758,719	692:30:04
$\Delta_5 \times \Delta_3$	442,472,050,753,920	25,606,173,722	1313:57:17 (M1Max8t)

Bonus Track I

Bonus Track I

For Raman

Triangulations with only simplices of min. vol.
of generalized hypersimplices [Manecke et al. 2020]:

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Triangulations with only simplices of min. vol.
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- $\Delta(6, 1, 4)$ has more than 249,295,320 (347,613 classes)
- $\Delta(6, 2, 4)$ has more than 7,248,961,080 (10,068,279 classes)

Bonus Track II

Bonus Track II

Other Results

Enumeration of (co-)circuits (different lex-min check):

Bonus Track II

Other Results

Enumeration of (co-)circuits (different lex-min check):

- $[0,1]^8$ has
  38,636,185,528,212,416 circuits in 3,858,105,362 classes
  (CPU: 163:37:00)
  (asked by Lisa Lamberti and Komei Fukuda)

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  (extends [Aichholzer & Aurnhammer 1996])

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Sideline

Necessary conditions for lex-extendability

Bonus Track II

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Necessary conditions for lex-extendability

- found for cocircuits

Bonus Track II

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Sideline

Necessary conditions for lex-extendability

- found for cocircuits
- but not so far for circuits.

Agenda

Conclusions/Questions

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Conclusions

Enumeration of orbits of triangulations accelerated by
“**Geometry meets Combinatorics**”:

Conclusions/Questions

Conclusions

Enumeration of orbits of triangulations accelerated by
"**Geometry meets Combinatorics**":

- critical-element tables for lex-min check

Conclusions/Questions

Conclusions

Enumeration of orbits of triangulations accelerated by "**Geometry meets Combinatorics**":

- critical-element tables for lex-min check
- minimal-element comparison for lex-extendability check

Conclusions/Questions

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Questions

Potential further research:

Conclusions/Questions

Conclusions

Enumeration of orbits of triangulations accelerated by
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- critical-element tables for lex-min check
- minimal-element comparison for lex-extendability check

Questions

Potential further research:

- Investigate the complexity of symLSRS.

Conclusions/Questions

Conclusions

Enumeration of orbits of triangulations accelerated by "**Geometry meets Combinatorics**":

- critical-element tables for lex-min check
- minimal-element comparison for lex-extendability check

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Potential further research:

- Investigate the complexity of symLSRS.
- Apply symLSRS to more examples.

Conclusions/Questions

Conclusions

Enumeration of orbits of triangulations accelerated by
"**Geometry meets Combinatorics**":

- critical-element tables for lex-min check
- minimal-element comparison for lex-extendability check

Questions

Potential further research:

- Investigate the complexity of symLSRS.
- Apply symLSRS to more examples.
- Represent flip-graph exploration in terms of subsets.