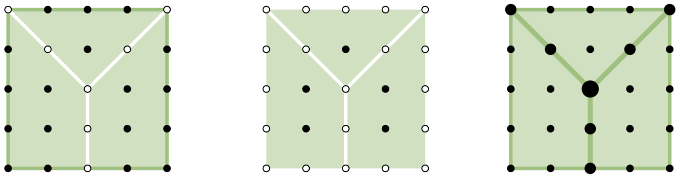
Pruned inside-out polytopes, combinatorial reciprocity theorems, and generalized permutahedra

Sophie Rehberg,
joint work (in progress) with Matthias Beck

Geometry meets Combinatorics in Bielefeld 2022



Motivation

Stanley (1973): For a graph g, $m \in \mathbb{Z}_{>0}$

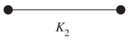
$$\chi_g(m) := \#\text{proper } m\text{-colorings of } g$$

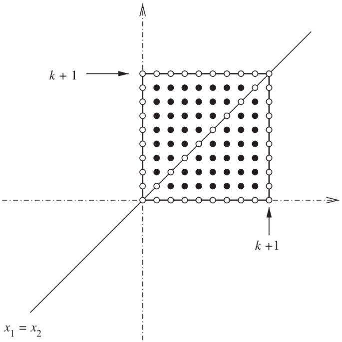
is a polynomial in m and

$$(-1)^d \chi_g(-m) = \#\text{pairs of compatible } m\text{-colorings and acyclic orientations.}$$

Inside-out polytopes (Beck-Zaslavsky 2006):

"polytope minus hyperplanes"





Ehrhart theory

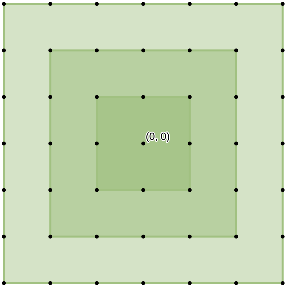
Let Q be a polytope in $\mathbb{R}^d$, $t \in \mathbb{Z}_{>0}$.

$$tQ := \{tx \in \mathbb{R}^d : x \in Q\}$$

$$\mathrm{ehr}_Q(t) := \#(tQ \cap \mathbb{Z}^d)$$

Example: square $Q = [-1, 1]^2$

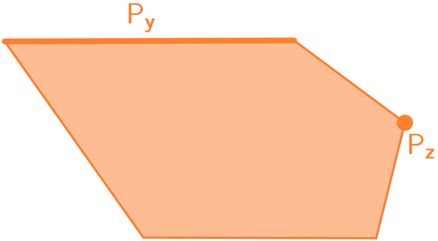
$$\mathrm{ehr}_{[-1,1]^2}(t) = (2t + 1)^2$$



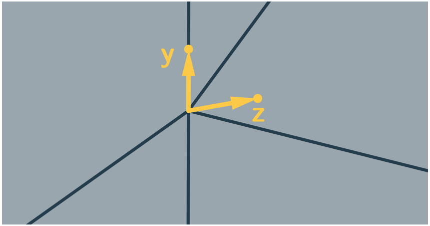
Theorem (Ehrhart, 1962)
For Q an integer polytope $\mathrm{ehr}_Q(t)$ agrees with a polynomial of degree $\dim(Q)$.

Theorem (Ehrhart-Macdonald reciprocity, 1971)
$(-1)^{\dim(Q)} \mathrm{ehr}_Q(-t) = \mathrm{ehr}_{Q^\circ}(t)$.

Polytopes and fans



polytope P in $\mathbb{R}^d$
$\mathrm{P}_{\mathbf{y}}$ maximal face
$\mathrm{P}_{\mathbf{z}}$ maximal vertex



normal fan $\mathcal{N}(\mathrm{P})$ in $(\mathbb{R}^d)^*$
$\mathbf{y} \in (\mathbb{R}^d)^*$ a direction
$\mathbf{z} \in (\mathbb{R}^d)^*$ a generic direction

$$N_{\mathrm{P}}(F) \coloneqq \{y \in (\mathbb{R}^d)^* \ : \ \mathrm{P}_y \supseteq F\}$$

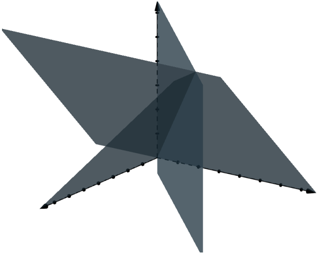
Polyhedral fans

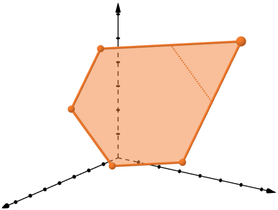
For a complete fan $\mathcal{N}$ in $\mathbb{R}^d$ define the **codimension-one fan** $\mathcal{N}^{\mathrm{co}\,1}$

$$\begin{aligned}\mathcal{N}^{\mathrm{co}\,1} &:= \{N \in \mathcal{N} : \ \operatorname{codim} N \geq 1\} \\ &= \{N \in \mathcal{N} : \ \dim N \leq d - 1\} .\end{aligned}$$

For a normal fan $\mathcal{N}(\mathrm{P})$ we get

$$\mathcal{N}^{\mathrm{co}\,1}(\mathrm{P}) = \{N_{\mathrm{P}}(F) \ : \ F \text{ a face of P with } \dim(F) \geq 1\}.$$



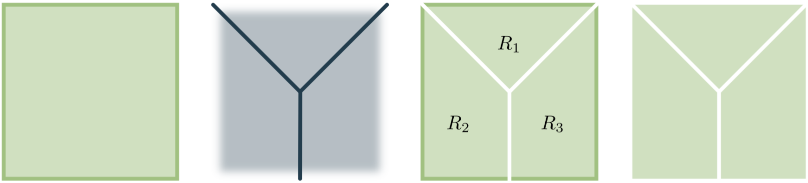


Pruned inside-out polytopes

For a polytope $\mathsf{Q} \subset \mathbb{R}^d$ and a complete fan $\mathcal{N}$ in $\mathbb{R}^d$ we call

$$\mathsf{Q} \setminus \left(\bigcup \mathcal{N}^{\mathrm{co}\,1} \right) = \biguplus_{\substack{N \in \mathcal{N}, \\ N \text{ full-dimensional}}} (\mathsf{Q} \cap N^\circ)$$

a **pruned inside-out polytope** and we call the connected components in $\mathsf{Q} \setminus \left(\bigcup \mathcal{N}^{\mathrm{co}\,1} \right)$ **regions**.

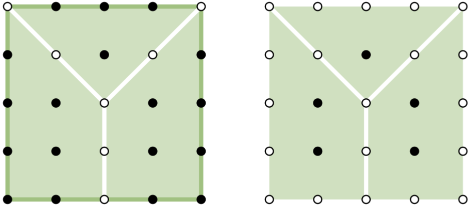


Q | $\mathcal{N}$ and $\mathcal{N}^{\mathrm{co}\,1}$ | $\mathsf{Q} \setminus \bigcup \mathcal{N}^{\mathrm{co}\,1}$ | $\mathsf{Q}^\circ \setminus \bigcup \mathcal{N}^{\mathrm{co}\,1}$

Pruned inside-out counting

For $t \in \mathbb{Z}_{>0}$ define the **inner pruned Ehrhart function** as

$$\mathrm{in}_{\mathsf{Q},\mathcal{N}^{\mathrm{co}\,1}}(t) := \# \left(t \cdot \left(\mathsf{Q} \setminus \bigcup \mathcal{N}^{\mathrm{co}\,1} \right) \cap \mathbb{Z}^d \right) .$$



$\mathrm{in}_{\mathsf{Q},\mathcal{N}^{\mathrm{co}\,1}}$ $\mathrm{in}_{\mathsf{Q}^\circ,\mathcal{N}^{\mathrm{co}\,1}}$

Note:

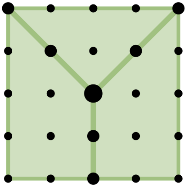
$$\mathrm{in}_{\mathsf{Q}^\circ,\mathcal{N}^{\mathrm{co}\,1}}(t) = \sum_{i=1}^{k} \mathrm{ehr}_{R_i^\circ}(t)$$

and it is a polynomial if regions R_i are integral.

Pruned inside-out counting

Define the **cumulative pruned Ehrhart function** for $t \in \mathbb{Z}_{>0}$ as

$$\mathrm{cu}_{\mathsf{Q},\mathcal{N}^{\mathrm{co}\,1}}(t) := \sum_{y \in t\mathsf{Q} \cap \mathbb{Z}^d} \#\left(N \in \mathcal{N},\ N \text{ full.dim.},\ y \in N\right).$$



$\mathrm{cu}_{\mathsf{Q},\mathcal{N}^{\mathrm{co}\,1}}$

Note:

$$\mathrm{cu}_{\mathsf{Q},\mathcal{N}^{\mathrm{co}\,1}}(t) = \sum_{i=1}^{k} \mathrm{ehr}_{\overline{R_i}}(t)$$

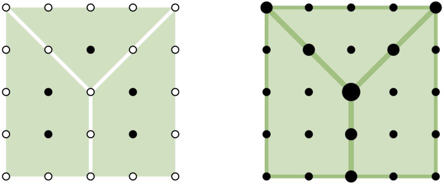
and it is a polynomial if regions R_i are is integral.

Theorem

For a polytope $\mathsf{Q} \subset \mathbb{R}^d$ and a complete fan $\mathcal{N}$ in $\mathbb{R}^d$ we have

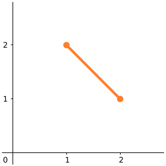
$$(-1)^{\dim \mathsf{Q}} \operatorname{in}_{\mathsf{Q}^\circ, \mathcal{N}^{\mathrm{co}\,1}}(-t) = \operatorname{cu}_{\mathsf{Q}, \mathcal{N}^{\mathrm{co}\,1}}(t)\,.$$

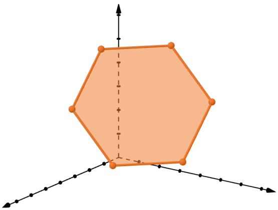
Proof.

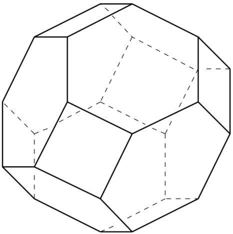


$$(-1)^d \operatorname{in}_{\mathsf{Q}^\circ, \mathcal{N}^{\mathrm{co}\,1}}(-t) = \sum_{i=1}^{k} (-1)^d \operatorname{ehr}_{R_i^\circ}(-t) = \sum_{i=1}^{k} \operatorname{ehr}_{\overline{R}_i}(t) = \operatorname{cu}_{\mathsf{Q}, \mathcal{N}^{\mathrm{co}\,1}}(t)\,.$$

Standard permutahedra (in type A)

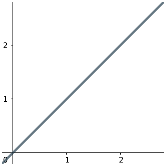


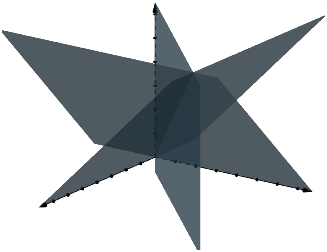




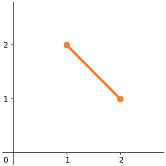
The **standard permutahedron** π_d is the convex hull of $d!$ vertices, namely, all the permutations of the point $(1, \ldots, d)$.

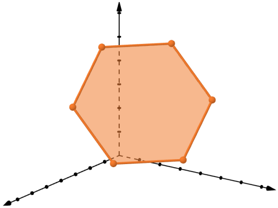
Braid fan and standard permutahedron (in type A)





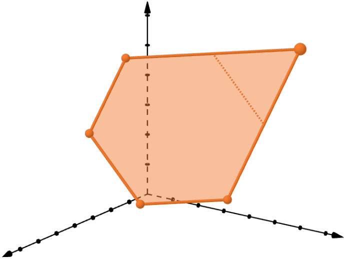
The **braid arrangement** $\mathcal{B}_d$ is the set of hyperplanes $H_{i,j} := \{x \in (\mathbb{R}^d)^* \colon x_i = x_j\}$. The **braid fan** is the fan induced by $\mathcal{B}_d$.

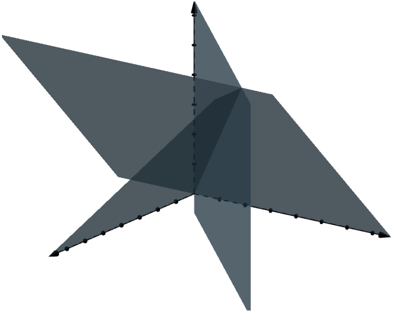




Generalized Permutahedra (in type A)

A polytope $P \subset \mathbb{R}^d$ is a **generalized permutahedron** if its normal fan $\mathcal{N}(P)$ is a coarsening of the braid fan.





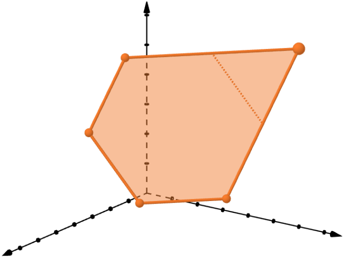
Graphically:
all deformations of standard permutahedron by translating facets.

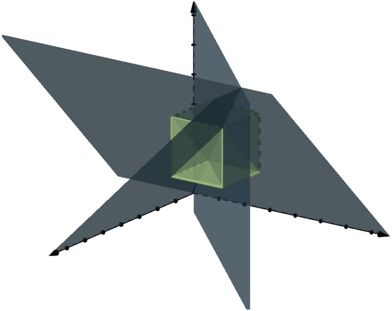
Reciprocity for generalized permutahedra (in type A)

Theorem (Aguiar, Ardila 2017; Billera, Jia, Reiner 2009)
For generalized permutahedra $\mathrm{P} \subset \mathbb{R}^d$, $m \in \mathbb{Z}_{>0}$

$$\chi_d(\mathrm{P})(m) := \# \left(\text{P-generic directions } y \in (\mathbb{R}^d)^* \text{ with } y \in [m]^d\right)$$

agrees with a polynomial in m of degree d.





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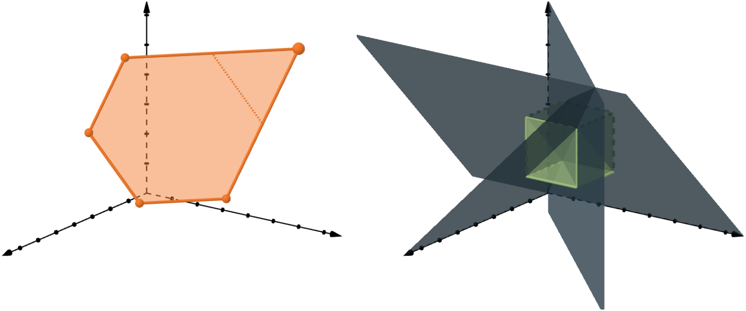
$$(-1)^d \chi_d(\mathrm{P})(-m) = \sum_{y \in [m]^d} \# \left(\text{vertices of } \mathrm{P}_y \right) .$$



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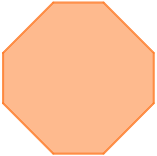
Special cases:

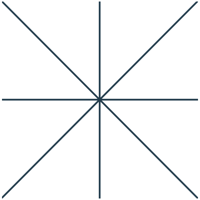
- Stanley's reciprocity theorem for **graphs**, Billera-Jia-Reiner's reciprocity theorem for **matroids**, Stanley's reciprocity theorem for **posets**, Bergman polynomial reciprocity for **matroids** (Aguiar, Ardila 2017)
- Aval-Karaboghossian-Tanasa's reciprocity theorem for **hypergraphs** (S.R. 2021+)

Type B generalized Coxeter permutahedra

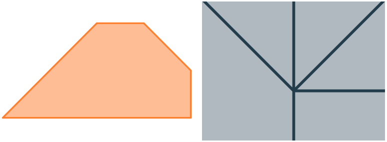
Type B Coxeter permutahedron

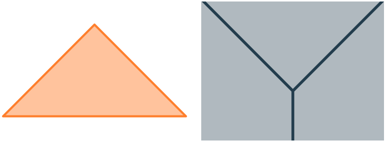


Type B Coxeter arrangement:
$\{x \in (\mathbb{R}^d)^* \colon x_i = \pm x_j\}$,
$\{x \in (\mathbb{R}^d)^* \colon x_i = 0\}$



Type B generalized Coxeter permutahedra:





Reciprocity in type B

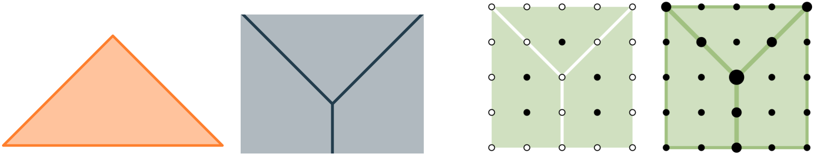
Theorem

For type B generalized Coxeter permutahedra $\mathrm{P} \subset \mathbb{R}^d$, $m \in \mathbb{Z}_{>0}$

$$\chi_d(\mathrm{P})(m) := \#\Big(\text{P-generic directions } y \in (\mathbb{R}^d)^* \quad \text{with } y \in \{-m, \ldots, -1, 0, 1, \ldots, m\}^d\Big)$$

agrees with a polynomial in m of degree d. Moreover,

$$(-1)^d \chi_d(\mathrm{P})(-m) = \sum_{y \in \{-m+1, \ldots, -1, 0, 1, \ldots, m-1\}^d} \#\left(\text{vertices of } \mathrm{P}_y\right) .$$



Final Remarks

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- Other combinatorial descriptions/applications?
- more general set-up possible $\rightarrow$ applications?

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Thank you for your attention!