Small topological counter-examples to the Hirsch Conjecture

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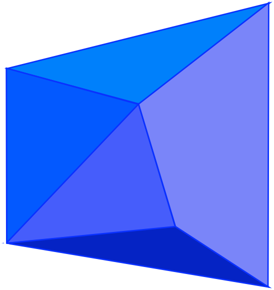
The graph of a polytope

Vertices and edges of a polytope *P* form a graph (finite, undirected)



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For example, $d(u, v) = 2$.

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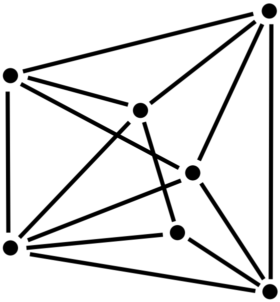

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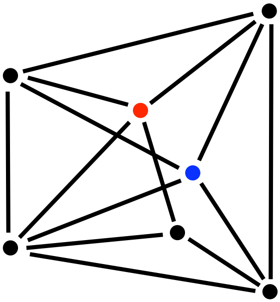

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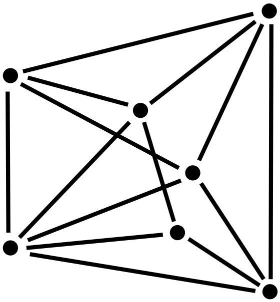

The (combinatorial) diameter of P is the maximum distance among its vertices:

$$\operatorname{diam}(P) = \max\{d(u,v) : u, v \in V(P)\}.$$

The Hirsch conjecture

Conjecture: Warren M. Hirsch (1957)

For every polytope P with n facets and dimension d,

$$\mathrm{diam}(P) \leq n - d.$$

polytope	facets	dimension	$n-d$	diameter
cube	6	3	3	3
dodecahedron	12	3	9	5
octahedron	8	3	5	2
k-prism	$k+2$	3	$k-1$	$\lfloor k/2 \rfloor + 1$
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Brief history of the conjecture

1. Communicated by W. M. Hirsch to G. Dantzig in 1957.
   Dantzig had recently invented the simplex method for LP.
2. Several special cases have been proved: $d \leq 3$, $n - d \leq 6$, $0/1$-polytopes, ...
3. But in the general case **we do not even know of a polynomial bound** for $\mathrm{diam}(P)$ in terms of n and d.
4. In 1967, Klee and Walkup found an unbounded counter-example.
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For the implications in linear programming , more important than the standard Hirsch conjecture is the following "polynomial version" of it:

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Theorem [Kalai-Kleitman 1992], "quasi-polynomial"

$$H(n,d) \leq n^{\log_2 d+2}, \qquad \forall n,d.$$

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The proofs of both are *surprisingly simple* and valid for objects much more general than polytopes.

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From polytopes to simplicial complexes

Definition

A d-polytope/polyhedron is simple if at every vertex exactly d facets meet. ($\simeq$ facet-defining hyperplanes are "in general position").

A d-polytope is simplicial if every facet has exactly d vertices. That is, if every proper face is a simplex. ($\simeq$ vertices are "in general position").

The (polar) dual of a simple polytope is simplicial, and vice-versa.

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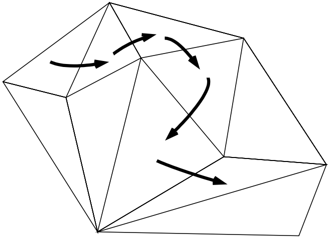
This suggests to pose the problem in the dual setting, for simplicial polytopes: We want to travel from one facet to another of a (simplicial) polytope Q along the “dual graph”, whose edges correspond to *ridges* of Q.



Regarded in this way the Hirsch question can be stated for more general objects, in various degrees of generality (from “simplicial spheres” to arbitrary “pure complexes”)

Normal simplicial complexes

In fact, both the Kalai-Kleitman bund and the Barnette-Larman bound hold for the following class of complexes:

Definition

A pure simplicial complex is called normal if the dual graph of every link is connected. (That is: you can go from any facet σ to any facet τ visiting only facets that contain $\sigma \cap \tau$)

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The importance of being normal

One may be tempted to extend the conjecture to arbitrary pure complexes, but it is relatively easy to find counter-examples.

In fact, we know *very precisely* the maximum diameter among all simplicial d-complexes with n vertices:

Theorem (Bohman-Newman 2022+)

Let $H_c(n, d)$ denote the maximum diameter among all pure d-complexes with n vertices. Then, for every d we have

$$H_c(n,d) \sim \frac{n^d}{d(d+1)!} \sim \frac{1}{d}\binom{n}{d+1}$$

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It is possible to go from u to v so that at each step we abandon a facet containing u and we enter a facet containing v.

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Proof: Let $H(n, d) = \max\{\delta(P) : P$ is a d-polytope with n facets$\}$. Then

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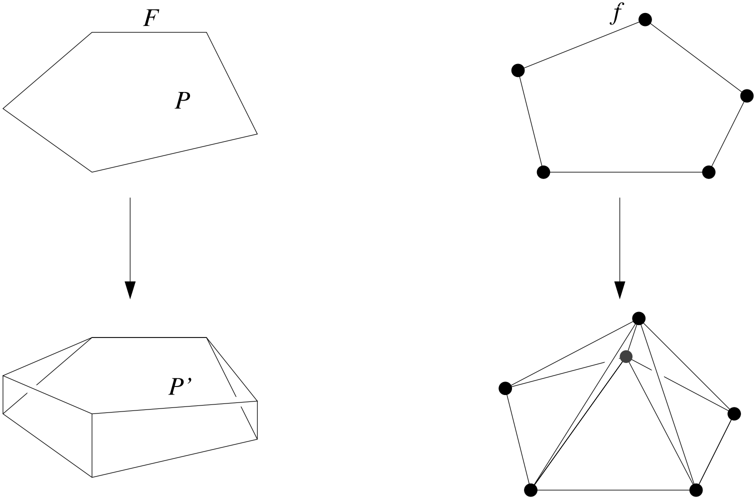

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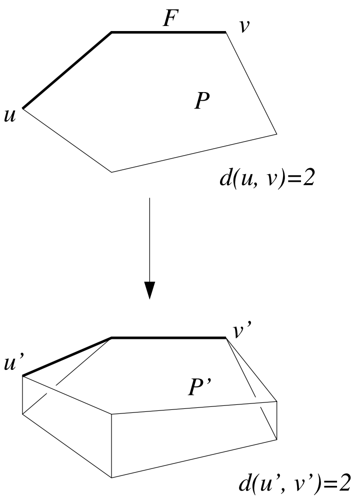

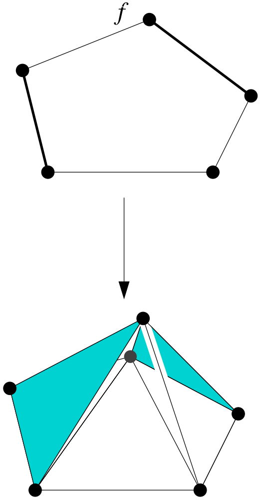

The counter-examples

The construction of counter-examples to the Hirsch conjecture has two ingredients:

1. A strong d-step theorem for prismatoids.
2. The construction of a prismatoid of dimension 5 and "width" 6.

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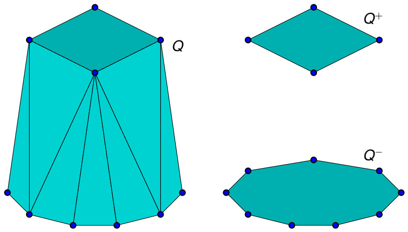
Prismatoids

Definition

A *prismatoid* is a polytope Q with two (parallel) facets Q^+ and Q^- containing all vertices.



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The *width* of a prismatoid is the *dual-graph* distance from Q^+ to Q^-.

Exercise

3-prismatoids have width ≤ 3.

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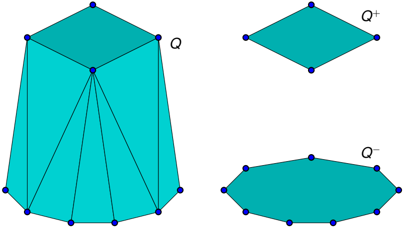

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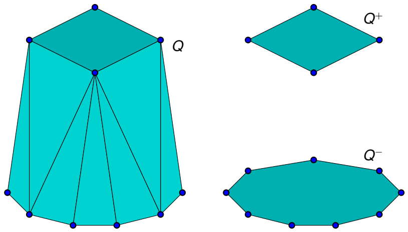

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Let Q be a prismatoid of dimension d, with $n > 2d$ vertices and width δ. Then there is another prismatoid Q' of dimension $d+1$, with $n+1$ vertices and width $\delta+1$.

That is: we can increase the dimension, width and number of vertices of a prismatoid, all by one, until $n = 2d$.

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In particular, if a prismatoid Q has width $> d$ then there is another prismatoid Q' (of dimension $n-d$, with $2n-2d$ vertices, and width $\geq \delta + n - 2d > n - d$) that violates (the dual of) the Hirsch conjecture.

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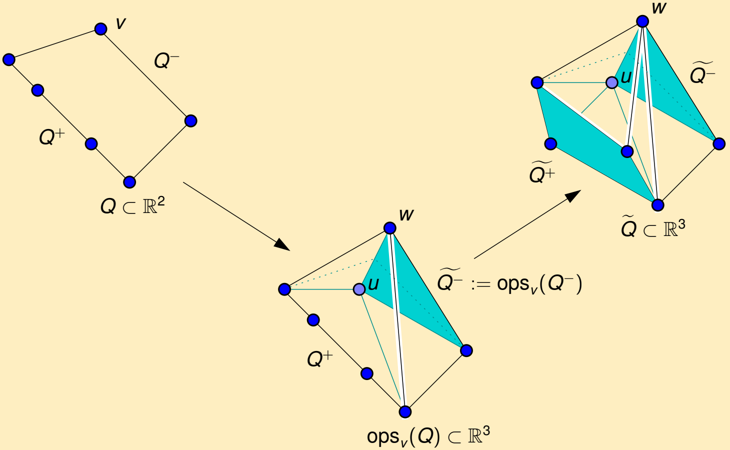
d-step theorem for prismatoids

Proof.



□

Width of prismatoids

So, to disprove the Hirsch Conjecture we only need to find a prismatoid of dimension d and width larger than d. *Its number of vertices and facets is irrelevant...*

Question

Do they exist?

- 3-prismatoids have width at most 3 (exercise).
- 4-prismatoids have width at most 4 [S.-Stephen-Thomas, 2011].
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Corollary

There is a 43-dimensional polytope with 86 facets and diameter (at least) 44.

Smaller 5-prismatoids of width > 5

With the same ideas

Theorem (Matschke-S.-Weibel, 2015)

The following prismatoid of dimension 5 and with 28 vertices has width 6.

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And with some more work:

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There is a 5-prismatoid with 25 vertices and of width 6.

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There is a non-Hirsch polytope of dimension 20 with 40 facets.

This one has been explicitly computed. It has $36,442$ vertices, and diameter 21.

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poly20dim21.ext

```
V-representation
begin
 40 21 rational
1  1        0     0    21     0        0        0        0           0            0            0            0            0       0         0         0         0          0          0           0
1  1        0     0    16    -8        0        0        0           0            0            0            0            0       0         0         0         1          1          0           0
1  1        0     0     0   -25        0        0        0           0            0            0            0            0       0         0         0         0          1          1           0
1  1        0     0   -16   -10        0        0        0           0            0            0            0            0       0         0         0         0          0          1           1
1  1        0     0   -21     0        0        0        0           0            0            0            0            0       0         0         0         0          0          0           1
1  1        0     0   -20     3        0        0        0           0            0            0            0            0       0         0         0         0          0          0           0
1  1        0     0     0    39        0        0        0           0            0            0            0            0       0         0         0         0          0          0           0
1  1        0     0    19     7        0        0        0           0            0            0            0            0       0         0         0         0          0          0           0
1  1    3/100 -1/50     0   -23        0        0        0           0            0            0            0            0       0         0         0         0          0          0           0
1  1   -3/100 -1/50     0    37        0        0        0           0            0            0            0            0       0         0         0         0          0          0           0
1  1  -3/2000 7/2000    0 388/10        0        0        0           0            0            0            0            0       0         1         0         0          0          0           0
1  1   3/2000 7/2000    0 -248/10 10000000 10000000 10000000 10000000000 100000000000 100000000000 100000000000 100000000000       1         0         0         0          0          0           0
1  1   3/2000 7/2000    0 -248/10 -10000000        0        0           0            0            0            0            0       1         0         0         0          0          0           0
1  1   3/2000 7/2000    0 -248/10 10000000 -10000000        0           0            0            0            0            0       1         0         0         0          0          0           0
1  1   3/2000 7/2000    0 -248/10 10000000 10000000 -10000000           0            0            0            0            0       1         0         0         0          0          0           0
1  1   3/2000 7/2000    0 -248/10 10000000 10000000 10000000 -10000000000            0            0            0            0       1         0         0         0          0          0           0
1  1   3/2000 7/2000    0 -248/10 10000000 10000000 10000000 10000000000 -100000000000            0            0            0       1         0         0         0          0          0           0
1  1   3/2000 7/2000    0 -248/10 10000000 10000000 10000000 10000000000 100000000000 -100000000000            0            0       1         0         0         0          0          0           0
1  1   3/2000 7/2000    0 -248/10 10000000 10000000 10000000 10000000000 100000000000 100000000000 -100000000000            0       1         0         0         0          0          0           0
1  1   3/2000 7/2000    0 -248/10 10000000 10000000 10000000 10000000000 100000000000 100000000000 100000000000 -100000000000       1         0         0         0          0          0           0
1 -1       30     0     0     0        0        0        1           1            0            0            0            0       0         0         0         0          0          0           0
1 -1        4   -15     0     0        0        0        0           1            1            0            0            0       0         0         0         0          0          0           0
1 -1        0 -33/2     0     0        0        0        0           0            1            1            0            0       0         0         0         0          0          0           0
1 -1       -1   -16     0     0        0        0        0           0            0            1            1            0       0         0         0         0          0          0           0
1 -1    -55/2     0     0     0        0        0        0           0            0            0            1            1       0         0         0         0          0          0           0
1 -1      -17    18     0     0        0        0        0           0            0            0            0            1       0         0         0         0          0          0           0
1 -1        0    38     0     0        0        0        0           0            0            0            0            0       0         0         0         0          0          0           0
1 -1       22    17     0     0        0        0        0           0            0            0            0            0       0         0         0         0          0          0           0
1 -1      -10     0   1/5  -1/5        0        0        0           0            0            0            0            0       0         0         0         0          0          0           0
1 -1 2999/100     0 -3/25  -1/5        0        0        1           0            0            0            0            0       0         0         0         0          0          0           0
1 -1 299999/10000 0     0 1/100        0        1        0           0            0            0            0            0       0         0         0         0          0          0           0
1 -1 -2745/100    0 1/5000 1/800        1        0        0           0            0            0            0            0       0         0         0         0          0          0           0
1 -1      -27     0 1/500 -1/80        0        0        0           0            0            0            0            0  100000  10000000  10000000  10000000  100000000  100000000  1000000000
1 -1      -27     0 1/500 -1/80        0        0        0           0            0            0            0            0 -100000         0         0         0          0          0           0
1 -1      -27     0 1/500 -1/80        0        0        0           0            0            0            0            0  100000 -10000000         0         0          0          0           0
1 -1      -27     0 1/500 -1/80        0        0        0           0            0            0            0            0  100000  10000000 -10000000         0          0          0           0
1 -1      -27     0 1/500 -1/80        0        0        0           0            0            0            0            0  100000  10000000  10000000 -10000000          0          0           0
1 -1      -27     0 1/500 -1/80        0        0        0           0            0            0            0            0  100000  10000000  10000000  10000000 -100000000          0           0
1 -1      -27     0 1/500 -1/80        0        0        0           0            0            0            0            0  100000  10000000  10000000  10000000  100000000 -100000000           0
1 -1      -27     0 1/500 -1/80        0        0        0           0            0            0            0            0  100000  10000000  10000000  10000000  100000000  100000000 -1000000000
end
allbases
printcobasis
```

Topological prismatoids

Definition

A $(d-1)$-dimensional *topological prismatoid* is a pure simplicial complex $\mathcal{C}$ homeomorphic to $\mathbb{S}_{d-2} \times [0,1]$, such that:

- All vertices lie on the boundary $\mathbb{S}_{d-2} \times \{0,1\} = B^+ \cup B^-$.
- The two boundary components B^+ and B^- are induced subcomplexes.

The *width* of $\mathcal{C}$ is two plus the minimum distance from a facet incident to B^+ to a facet incident to B^-.

For technical reasons we need to assume that the two bases of our topological prismatoids are polytopal. Let us call a topological prismatoid with polytopal bases *semipolytopal*.

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Theorem (Strong d-step theorem for topological prismatoids)

Let $\mathcal{C}$ be a semipolytopal prismatoid of dimension $d-1$, with $n > 2d$ vertices and width δ. Then there is another prismatoid $\widetilde{\mathcal{C}}$ of dimension d, with $n+1$ vertices and width $\delta+1$.

That is: we can increase the dimension, width and number of vertices of a semipolytopal prismatoid, all by one, until $n = 2d$.

Corollary

From any semipolytopal $(d-1)$-prismatoid $\mathcal{C}$ of width $> d$ one can construct a simplicial sphere (of dimension $n-d-1$, with $2n-2d$ vertices, and width $\geq \delta + n - 2d > n-d$) that violates the Hirsch bound.

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Let $\mathcal{C}$ be a semipolytopal prismatoid of dimension $d-1$, with $n > 2d$ vertices and width δ. Then there is another prismatoid $\widetilde{\mathcal{C}}$ of dimension d, with $n+1$ vertices and width $\delta+1$.

That is: we can increase the dimension, width and number of vertices of a semipolytopal prismatoid, all by one, until $n = 2d$.

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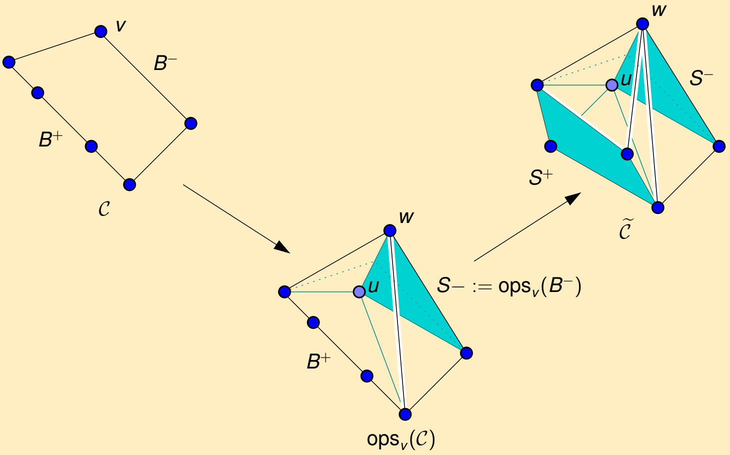
d-step theorem for prismatoids

Proof.



□

The technical difficulty

The first step in the proof (the one-point suspension on the "facet" B^-) works without problem, since ops is a topological operation.

For the second step, we need to increase the dimension of the "facet" B^+ adding no vertices. That is, we need that boundary component of the prismatoid (which is a $d-2$-sphere) to be embedded in a simplicial $(d-1)$-sphere without new vertices.

Question ("0-point suspension of spheres")

Can every simplicial k-sphere with more than $k+2$ vertices be embedded in a $(k+1)$-sphere with no extra vertices?

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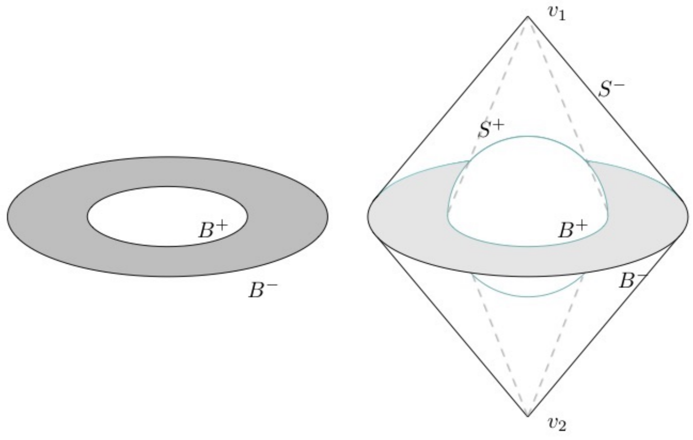
d-step theorem for prismatoids

Topological perspective.



□

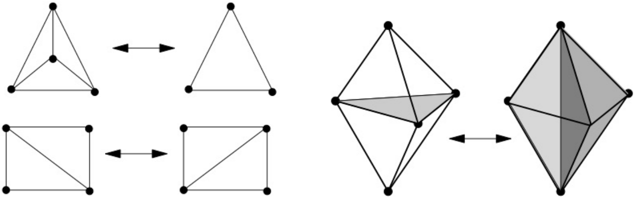
Flipping in simplicial manifolds

Definition

A *bistellar flip* in a simplicial $d-1$-manifold $\mathcal{C}$ is a pair (f, l) of pairwise disjoint subsets of vertices such that f is a face, l is a minimal nonface, and $\mathrm{lk}_{\mathcal{C}}(f) = \partial(l)$ (this implies $|f| + |l| = d + 2$).

The result of the flip is the complex

$$\mathcal{C}' = \mathcal{C} \setminus \mathrm{st}_{\mathcal{C}}(f) \cup (l * \partial(f)).$$



Flips in prismatoids

Observe that a bistellar flip can remove a vertex (if f is a single vertex) or insert a vertex (if I is empty and f is a facet).

In a topological prismatoid we can do two types of flips:

1. Interior flips, defined just as above, with the requirement that I intersects the two boundary components of $\mathcal{C}$ (so that after the flip the boundary components are still induced subcomplees).
2. Boundary flips: these are flips in one of the boundary components (which is itself a manifold). We now require all facets of $\mathrm{st}_{\mathcal{C}}(f) = f * \partial(I)$ to be coned to the same vertex, and after the flip we cone $I * \partial(f)$ to that same vertex.

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Simulated annealing

Flips allow us to explore the “space” of non-Hirsch topological prismatoids: starting with one of the non-Hirsch (polytopal) prismatoids of dimension four we do random flips and check whether the new prismatoids are still non-Hirsch.

In order to (try to) get smaller prsimatoids we flip with a simulated annealing strategy, that favours flips “in the right direction”.

Favoring only flips that remove vertices is not a good strategy: most prismatoids do not have such flips.

What we do is to use as cost function a generalized mean of the number of neighbors of all vertices in the prismatoid, trying to produce vertices with few neighbors. Vertex-removing flips happen exactly one a vertex has only $d + 1$ neighbors.

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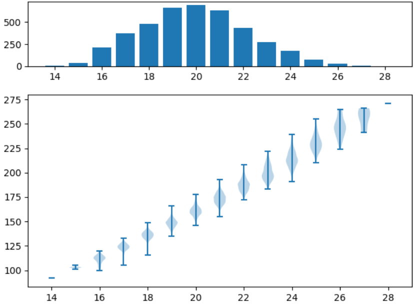
The result

We ran our algorithm for three days. We completed 4093 runs, all starting with a prismatoid with 28 vertices. This gave us 4093 non-Hirsch topological 4-prismatoids, with number of vertices ranging between 14 and 28:



Top: number of prismatoids by nbr. of vertices.

Bottom: distribution of vertices vs. facets

The result

In particular, we obtained four 4-dimensional non-Hirsch topological prismatoids with 14 (=7+7) vertices. Thus:

Theorem

There exist 8-dimensional spheres with 18 vertices that violate the Hirsch bound.

We have checked that these four are shellable, but Pfeifle (2020+) and Gouveia-Macchia-Thomas (2023) have proved that they are not polytopal.

In fact, they have found non-polytopality certificates also for all our prismatoids with 15 vertices, and for some with more vertices.

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The result

For the record, here is a non-Hirsch topological prismatoid with 14 vertices:

0256g	126cg	123ae	025af	14abf	26abf	25abf	5acde
0245f	015bg	124af	025bg	13abf	26ace	25abe	5abde
1256g	015bf	123af	024af	13acd	26abe	01bcg	3bcfg
1245f	014ae	123bf	024ae	13acg	05abf	06bcg	3acfg
0234e	013ae	123bg	025ae	13abg	03ace	26bcg	3abfg
0123d	013ad	123cd	025be	06bce	05ace	23bcg	0bcde
1234e	016cd	123cg	125bg	04abd	03acd	23bcf	6bcef
0126d	016cg	026bg	125bf	04bcd	05acd	23acf	6acef
0156g	014bf	026cd	023ce	14bcg	05cde	23ace	6abef
0145f	014ad	026ce	023cd	14abg	05bde	26bcf	4abcg
0134e	014cd	026be	124ae	14acg	05abd	26acf	4abcd
	014bc	126cd		14acd	04abf		

For more details see

Francisco Criado, Francisco Santos.
Topological Prismatoids and Small Simplicial Spheres of Large Diameter.
Experimental Mathematics, 31:2 (2022), 461–473.
DOI: 10.1080/10586458.2019.1641766.
arXiv:1807.03030

T H A N K Y O U