On Brauer's height zero conjecture

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p : prime number
k : $\bar{\mathbb{F}}_p$
$\mathfrak{p}$: maximal ideal of complete d.v.r. in characteristic 0 lifting $\bar{\mathbb{F}}_p$
G : finite group
$\mathrm{Irr}(G)$: $\{$complex irreducible characters of $G\}$
Block decomposition of kG into indecomposable algebra factors:

$$kG = \prod_{B \in Bl_p(G)} B$$

$$\updownarrow$$

$$\mathrm{Irr}(G) = \bigsqcup_{B \in Bl_p(G)} \mathrm{Irr}(B)$$

- χ, χ' in same p-block iff

$$\frac{|G|\chi(x)}{|C_G(x)|\chi(1)} - \frac{|G|\chi'(x)}{|C_G(x)|\chi'(1)} \in \mathfrak{p}$$

for all $x \in G$.

$$kG = \prod_{B \in Bl_p(G)} B, \quad \mathrm{Irr}(G) = \bigsqcup_{B \in Bl_p(G)} \mathrm{Irr}(B)$$

Definition

Let Q be a p-subgroup of G. The Brauer homomorphism $\mathrm{Br}_Q : kG \to kC_G(Q)$ is defined by $\sum_{g \in G} \alpha_g g \to \sum_{g \in C_G(Q)} \alpha_g g$.

Definition

A defect group of B is a p-subgroup P of G satisfying one of the following equivalent conditions:

- *P maximal s.t. $\mathrm{Br}_P(1_B) \neq 0$.*
- *P maximal s.t. for some p-regular $x \in C_G(P)$, $\frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}(B)} \chi(1)\chi(x) \notin \mathfrak{p}$.*
- *P minimal s.t. B is a summand of $B \otimes_{kP} B$ as (B, B)-bimodules.*

• Defect groups exist and are unique upto G-conjugacy.

B: p-block of G, P: defect group of B.
Representation theory of B is influenced by P, e.g.:

• (Brauer) $|P| = max \left\{ \frac{|G|_p}{\chi(1)_p} \; : \; \chi \in \mathrm{Irr}(B) \right\}$.

• (Brauer-Feit) $|\mathrm{Irr}(B)| \leq \frac{1}{4}|P|^2 + 1$.

• (Brauer) $P = 1$ iff $B = \mathrm{Mat}_n(k)$ iff $|\mathrm{Irr}(B)| = 1$ iff $\exists\ \chi \in \mathrm{Irr}(B)$ with $\chi(1)_p = |G|_p$.

⋮

⋮

And many conjectures...

B: p-block of G, P: defect group of B.

• $|P| = max \left\{ \frac{|G|_p}{\chi(1)_p} \ : \ \chi \in \mathrm{Irr}(B) \right\}$.

Brauer's height zero conjecture, 1955

P *abelian* $\iff \forall \chi \in \mathrm{Irr}(B),\ |P| = \frac{|G|_p}{\chi(1)_p}$.

Will refer to the forward direction of conjecture as (HZ1), reverse direction as (HZ2).
Brauer's evidence (1963):
• $(HZ1)$ true if either P cyclic (Brauer) or G p-solvable (Fong).
• $(HZ2)$ true if G p-solvable and B principal block (Fong).
Many cases handled subsequently.

• Structural explanation for $(HZ1)$ is provided by Broue's Abelian Defect Group Conjecture (1990):
If P abelian, then B is derived equivalent to a p-block of kH, with $H = O_{p',p,p'}(H)$, and $O_{p'}(H) \leq Z(H)$, P a Sylow p-subgroup of H.
[formulation above depends on a theorem of Külshammer.]

B: p-block of G, P: defect group of B.

Conj. (HZ1): If P abelian, then $\forall \chi \in \mathrm{Irr}(B),\ |P| = \frac{|G|_p}{\chi(1)_p}$.
Conj. (HZ2) : Converse.

Theorem (Berger-Knörr, 1988)

$(HZ1)$ *true if true for quasi-simple groups.*

Theorem (2011)

$(HZ1)$ *true.*

To get from the reduction to the final theorem, need to solve:

Problem

For all finite quasi-simple G, and all primes p, "describe" the p-blocks of G and their defect groups.

G : quasi-simple, $\bar{G} = G/Z(G)$.

- $\bar{G}$ sporadic: Use character tables.
- $\bar{G} = A_n$:

$$\left.\begin{array}{lcl} \mathrm{Irr}(S_n) & \leftrightarrow & \{\text{partitions of } n\} \\ \chi_\lambda & \leftrightarrow & \lambda \\ \chi_\lambda(1) & = & \dfrac{n!}{\text{product of hook lengths of } \lambda} \end{array}\right\} \text{(Schur,1911)}$$

$$\left.\begin{array}{lcl} Bl_p(S_n) & \leftrightarrow & \{p-\text{cores of partitions of } n\} \\ B_\tau & \leftrightarrow & \tau \\ \chi_\lambda \in \mathrm{Irr}(B_\tau) & \text{iff} & \tau = p-\text{core of } \lambda \end{array}\right\} \text{Robinson-Brauer '47}$$

If $|\tau| = m$, Sylow $p-$subgroup of S_{n-m} is a defect group of B_τ.

-Similar (but independent, and more complicated) combinatorial story for projective representations of S_n (Schur, Morris, Humphreys).

(HZ) True for G if $\bar{G} = A_n$. (Olsson '90)

• $\bar{G}$: finite group of Lie type in characteristic.
If p is the characteristic of $\bar{G}$, then few p-blocks (Humphreys, 1971):

$$\begin{array}{lcl} Bl_p(G) & \leftrightarrow & \mathrm{Irr}(Z(G)) \cup \{\text{Steinberg character}\} \\ \text{Defect groups} & : & \text{Sylow}\, p\!-\!\text{subgps}\,,\{1\} \end{array}$$

Remaining case: p different from the characteristic of $\bar{G}$.

Conceptual set up:
$\mathbf{G}$: simple algebraic group over $\bar{\mathbb{F}}_q$, q a prime power.
$F : \mathbf{G} \to \mathbf{G}$, a Steinberg endomorphism w.r.t. $\mathbb{F}_q$.
$G = \mathbf{G}^F$.

Dual set up:
$\mathbf{G}^*$: dual group,
$F^* : \mathbf{G}^* \to \mathbf{G}^*$ compatible Steinberg,
$G^* = {\mathbf{G}^*}^{F^*}$.

$G = \mathbf{G}^F, \quad G^* = {\mathbf{G}^*}^{F^*}$.
Lusztig induction:
L an F-stable Levi subgroup of some parabolic subgroup of **G**

$$\mathrm{R}_{\mathbf{L}}^{\mathbf{G}} : \mathbb{Z}\mathrm{Irr}(L) \to \mathbb{Z}\mathrm{Irr}(G), \quad (L = \mathbf{L}^F).$$

$$^*\mathrm{R}_{\mathbf{L}}^{\mathbf{G}} : \mathbb{Z}\mathrm{Irr}(G) \to \mathbb{Z}\mathrm{Irr}(L), \quad \text{adjoint map.}$$

The definition of $\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}$ is geometric. A special case is Harish-Chandra induction:
If **L** is a Levi of an *F-stable* parabolic **P** of **G**, then

$$\mathrm{R}_{\mathbf{L}}^{\mathbf{G}} = \mathrm{Ind}_P^G \circ \mathrm{Inf}_L^P, \quad (P = \mathbf{P}^F).$$

Lusztig's theory of characters (80's)

- $$\mathrm{Irr}(G) = \bigsqcup_{s \in G^*_{ss}/\sim} \mathcal{E}(G,(s)).$$

(union is over conjugacy classes of semisimple elements of G^*)

Definition

$\mathcal{E}(G,(s))$: Lusztig series associated to s.
$\mathcal{E}(G,1)$: Unipotent characters of G.

- $\mathcal{E}(G,1)$ is parametrised independently of q — depends only on the type of $(\mathbf{G},F)$. [e.g. If $\mathbf{G} = GL_n$, then $\mathcal{E}(G,1) \leftrightarrow \mathrm{Irr}(S_n)$]

- For any $s \in G^*_{ss}$, there is a bijection

$$\Psi_s : \mathcal{E}(G,(s)) \to \mathcal{E}(C_{G^*}(s),1)$$

such that for all $\chi \in \mathcal{E}(G,(s))$

$$\chi(1) = \Psi_s(\chi)(1)|G^* : C_{G^*}(s)|_{q'}$$

Blocks

$G = \mathbf{G}^F = G(q), \quad (p, q) = 1.$

• (Fong-Srinivasan, 1982): Description of p-blocks of finite general linear and unitary groups.
[Conj. (HZ1) true if $\mathbf{G}$ is of type A, (Blau-Ellers, 1999).]

For s a semisimple p'- element of G^*, set

$$\mathcal{E}_p(G, (s)) := \bigsqcup_{t \in C_{G^*}(s)_p/\sim} \mathcal{E}(G, (ts)).$$

• (Broué-Michel, 1989) $\mathcal{E}_p(G, (s))$ is a union of p-blocks.

• (Hiss, 1989) If B is a p-block in $\mathcal{E}_p(G, (s))$, then $\mathrm{Irr}(B) \cap \mathcal{E}(G, (s)) \neq \emptyset$.
Our problem reduces to: For all p-regular semisimple $s \in G^*$, determine p-blocks and defect groups in $\mathcal{E}_p(G, (s))$.
Solution: Nice fit between Brauer and Lusztig theories.

$G = \mathbf{G}^F = G(q)$, $(p,q) = 1$, $s \in G^*_{ss}$, $p \not| o(s)$.
B : p-block of G in $\mathcal{E}_p(G,(s))$.

$\mathbf{L}$: F-stable Levi subgroup of $\mathbf{G}$ with $s \in L^*$, $L = \mathbf{L}^F$.
C: p-block of L in $\mathcal{E}_p(G,(s))$.
λ : irreducible character of L in $C \cap \mathcal{E}(L,(s))$.
$Z = Z(L)_p$.

$\mathrm{R}_{\mathbf{L}}^{\mathbf{G}} : \mathbb{Z}\mathrm{Irr}(L) \to \mathbb{Z}\mathrm{Irr}(G)$, Lusztig induction.
$\mathrm{Br}_Z : kG \to kC_G(Z)$, Brauer homomorphism.

Theorem (Cabanes)

Suppose that $\mathbf{L} = C_{\mathbf{G}}(Z)$ *and* $\lambda(1) = |L:Z|_p$. *Then,*

$$\mathrm{Br}_Z(1_B)1_C \neq 0 \iff \text{the constituents of } \mathrm{R}_{\mathbf{L}}^{\mathbf{G}}(\lambda) \text{ lie in } B.$$

Further, if $\mathrm{Br}_Z(1_B)1_C \neq 0$ *and the relative Weyl group* $N_G(\mathbf{L},\lambda)/L$ *is a* p'*-group, then* Z *is a defect group of* B.

d-Harish-Chandra theory

$d \in \mathbb{N}$, $\Phi_d(x)$: d-th cyclotomic polynomial.
d-split Levi subgroups: centralisers in $\mathbf{G}$ of F-stable tori $\mathbf{T}$ with $|\mathbf{T}^F| = \Phi_d(q)^m$ (some m).
$\chi \in \mathrm{Irr}(G)$ is *d-cuspidal* if

$$\langle \chi, \mathrm{R}_{\mathbf{L}}^{\mathbf{G}}(\psi) \rangle = 0 \text{ for all proper } d-\text{split } \mathbf{L} < \mathbf{G},\ \psi \in \mathrm{Irr}(L).$$

A d-cuspidal pair is a pair $(\mathbf{L}, \lambda)$ such that $\mathbf{L}$ is d-split and λ is a d-cuspidal character of L.

Theorem (Broué-Malle-Michel, 1993)

Let d be the order of q modulo p. Suppose that p is sufficiently large and $s = 1$.

- *If (L, λ) is a unipotent d-cuspidal pair, then*

  $$\mathbf{L} = C_{\mathbf{G}}(Z) \quad \textit{and} \quad \lambda(1) = |L : Z|_p,$$

  where $Z = Z(L)_p$.
- $\{$ *blocks* $\} \overset{1-1}{\to} \{$*d-cuspidal pairs*$\}$.

$G = \mathbf{G}^F = G(q), \;\; (p, q) = 1$, $s \in G^*_{ss}$, $p \not| o(s)$.
The Broué- Malle-Michel situation, i.e., $s = 1$ and p large may be considered as the ”generic case”. Carries over (with modifications) to the other cases:

• $s = 1$, p good for $\mathbf{G}$, odd (Cabanes-Enguehard, 1994)

• p good, odd (Cabanes-Engeuhard, 1999) [special cases - (Fong-Srinivasan)]

• $s = 1$, p bad (Enguehard, 2000)

• $p = 2$, G classical (Enguehard, 2008) [special cases- (An)]

Remaining Case:
• p bad, G exceptional, $s \neq 1$

Theorem (Bonnafé-Rouquier, 2003)

Suppose that $\mathbf{L}$ *an* F*-stable Levi subgroup of* $\mathbf{G}$ *with* $C_{\mathbf{G}^*}(s) \leq \mathbf{L}^*$*. Then,* $\mathrm{R}_{\mathbf{L}}^{\mathbf{G}}$ *induces a Morita equivalence between* p*-blocks of* L *in* $\mathcal{E}_p(L, (s))$ *and* p*-blocks of* G *in* $\mathcal{E}_p(G, (s))$*.*

May assume that s is *quasi-isolated* , i.e., that $C_{\mathbf{G}^*}(s)$ is not contained in any proper Levi subgroup of $\mathbf{G}^*$.

• p bad, s quasi-isolated, G exceptional (K-Malle, 2011).
[special cases-(Schwewe, Deriozitis-Michler, Hiss, Ward, Malle]

So, now have a parametrization of p-blocks (and defect groups) of G, for all p, all quasi-simple G.
Getting from the parametrization to Conjecture (HZ1) required a bit more work. For instance:

Theorem (K-Malle, 2011)

If $\mathbf{G}$ *is simple and simply connected, then Bonnafé-Rouquier Morita equivalences preserve abelian defect groups.*