Weakly commensurable groups, with applications to differential geometry

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(joint work with Gopal Prasad)

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Outline

1. Geometric introduction
   - Isospectral and length-commensurable manifolds
   - Hyperbolic manifolds
2. Weakly commensurable arithmetic groups
   - Definition of weak commensurability
   - Arithmetic groups
   - Results on weak commensurability
3. Back to geometry
   - Length-commensurability vs. weak commensurability
   - Some results

References

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[2] — , —, *Local-global principles for embedding of fields with involution into simple algebras with involution*, Comment. Math. Helv. **85**(2010), 583-645.

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SURVEY:

[4] — , —, *Number-theoretic techniques in the theory of Lie groups and differential geometry*, 4th International Congress of Chinese Mathematicians, AMS/IP Stud. Adv. Math. **48**, AMS 2010, pp. 231-250.

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M_1 and M_2 are commensurable if they have a common finite-sheeted cover:

$$\begin{array}{ccc} & M & \\ \swarrow & & \searrow \\ M_1 & & M_2 \end{array}$$

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Example: Let M_1 and M_2 be *spheres* of radii r_1 and r_2. Then $L(M_i) = \{2\pi r_1\}$. So, $L(M_1) = L(M_2) \Rightarrow M_1$ & M_2 are isometric.

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Even though there are examples of noncommensurable isospectral manifolds (Lubotzky et al.), it appears that commensurability is the property that one may be able to establish in various situations.

Conditions (1) & (2) are not invariant under passing to a commensurable manifold, while condition (3) - length-commensurability $(\mathbb{Q} \cdot L(M_1) = \mathbb{Q} \cdot L(M_2))$ - is.

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Conditions (1), (2) and (3) are related:

- For Riemann surfaces: $\mathcal{E}(M_1) = \mathcal{E}(M_2) \Leftrightarrow \mathcal{L}(M_1) = \mathcal{L}(M_2)$
- For *any* compact locally symmetric spaces:

$$\mathcal{E}(M_1) = \mathcal{E}(M_2) \Rightarrow L(M_1) = L(M_2).$$

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So, results for length-commensurable locally symmetric spaces imply results for isospectral spaces.

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In particular:

- we know **when** length-commensurability $\Rightarrow$ commensurability (answer depends on Lie type of isometry group)
- locally symmetric spaces **length-commensurable** to a given arithmetically defined locally symmetric space form finitely many commensurability classes.

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Arithmetically defined hyperbolic d-manifold is $M = \mathbb{H}^d/\Gamma$,
where Γ is an *arithmetic* subgroup of $\mathcal{G}$.

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(1) *Suppose d is even or $\equiv 3 \pmod 4$.*

If M_1 and M_2 are not commensurable then after a possible interchange of M_1 and M_2, there exists $\lambda_1 \in L(M_1)$ such that for any $\lambda_2 \in L(M_2)$, the ratio λ_1/λ_2 is transcendental.

In particular, *M_1 and M_2 are not length-commensurable.*

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Under minor additional conditions we prove the following:

Let $\mathcal{F}_i$ *be subfield of* $\mathbb{R}$ *generated by* $L(M_i)$. *Then* $\mathcal{F}_1\mathcal{F}_2$ *has infinite transcendence degree over* $\mathcal{F}_1$ *or* $\mathcal{F}_2$.

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(We have similar results for complex and quaternionic hyperbolic spaces.)

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Weak commensurability

Let G_1 and G_2 be two semi-simple groups over a field F of *characteristic zero*.

- Semi-simple $g_i \in G_i(F)$ $(i = 1, 2)$ are weakly commensurable **if** there exist maximal F-tori $T_i \subset G_i$ such that $g_i \in T_i(F)$ and for some $\chi_i \in X(T_i)$ (defined over $\overline{F}$) we have

$$\chi_1(g_1) = \chi_2(g_2) \neq 1.$$

- (Zariski-dense) subgroups $\Gamma_i \subset G_i(F)$ are weakly commensurable **if** every semi-simple $\gamma_1 \in \Gamma_1$ of infinite order is weakly commensurable to some semi-simple $\gamma_2 \in \Gamma_2$ of infinite order, and vice versa.

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Recall: given an *F-torus* $T \subset \mathrm{GL}_n$, an *element* $t \in T(F)$, and a *character* $\chi \in X(T)$, the *character value*

$$\chi(t) = \lambda_1^{a_1} \cdots \lambda_n^{a_n}$$

where $\lambda_1, \ldots, \lambda_n$ are the *eigenvalues* of t (i.e. t is *conjugate* to $\mathrm{diag}(\lambda_1, \cdots, \lambda_n)$), and $a_1, \ldots, a_n \in \mathbb{Z}$.

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Let $g_1 \in G_1(F)$ and $g_2 \in G_2(F)$ be semi-simple elements with eigenvalues

$$\lambda_1, \ldots, \lambda_{n_1} \quad \text{and} \quad \mu_1, \ldots, \mu_{n_2}.$$

Then g_1 and g_2 are weakly commensurable if

$$\chi_1(g_1) = \lambda_1^{a_1} \cdots \lambda_{n_1}^{a_{n_1}} = \mu_1^{b_1} \cdots \mu_{n_2}^{b_{n_2}} = \chi_2(g_2) \neq 1$$

for some $a_1, \ldots a_{n_1}$ and $b_1, \ldots b_{n_2} \in \mathbb{Z}$.

Example

Let

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1/6 \end{pmatrix} \ , \ B = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/6 \end{pmatrix} \ \in \ SL_3(\mathbb{C}).$$

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However, **no** powers A^m and B^n $(m, n \neq 0)$ are conjugate.

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MAIN QUESTION: *What can one say about Zariski-dense subgroups* $\Gamma_i \subset G_i(F)$ $(i = 1, 2)$ *given that they are weakly commensurable?*

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RECALL: subgroups $\mathcal{H}_1$ and $\mathcal{H}_2$ of a group $\mathcal{G}$ are commensurable if

$$[\mathcal{H}_i : \mathcal{H}_1 \cap \mathcal{H}_2] < \infty \quad \text{for} \quad i = 1,2.$$

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Γ_1 and Γ_2 are commensurable up to an F-isomorphism between G_1 and G_2 if there exists an F-isomorphism $\sigma \colon G_1 \to G_2$ such that

$$\sigma(\Gamma_1) \quad \text{and} \quad \Gamma_2$$

are commensurable in usual sense.

Algebraic Perspective

GENERAL FRAMEWORK: *Characterization of linear groups in terms of spectra of its elements.*

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COMPLEX REPRESENTATIONS OF FINITE GROUPS:

Let Γ be a finite group,

$$\rho_i \colon \Gamma \to GL_{n_i}(\mathbb{C}) \quad (i = 1, 2)$$

be representations. Then

$$\rho_1 \simeq \rho_2 \quad \Leftrightarrow \quad \chi_{\rho_1}(g) = \chi_{\rho_2}(g) \quad \forall g \in \Gamma,$$

where $\chi_{\rho_i}(g) = \operatorname{tr} \rho_i(g) = \sum \lambda_j \quad (\lambda_1, \ldots, \lambda_{n_i}$ eigenvalues of $\rho_i(g))$

Algebraic perspective

- Data afforded by weak commensurability is more convoluted than data afforded by character of a group representation: when computing
$$\chi(g) = \lambda_1^{a_1} \cdots \lambda_n^{a_n}$$
one can use *arbitrary* integer weights $a_1, \ldots, a_n$. **So,** weak commensurability appears to be more difficult to analyze.

- **Example.** Let $\Gamma \subset SL_n(\mathbb{C})$ be a neat Zariski-dense subgroup. For $d > 0$, let
$$\Gamma^{(d)} = \langle \gamma^d \mid \gamma \in \Gamma \rangle.$$
Then any $\Gamma^{(d)} \subset \Delta \subset \Gamma$ is weakly commensurable to Γ.

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Then any $\Gamma^{(d)} \subset \Delta \subset \Gamma$ is weakly commensurable to Γ.
So, one needs to limit attention to some special subgroups in order to generate meaningful results.

Geometric perspective

- Weak commensurability (of fundamental groups) adequately reflects length-commensurability of locally symmetric spaces.

- Let $G = SL_2$. Corresponding symmetric space:

$$SO_2(\mathbb{R})\backslash SL_2(\mathbb{R}) = \mathbb{H} \quad \text{(upper half-plane)}$$

- Any (compact) Riemann surface of genus > 1 is of the form

$$M = \mathbb{H}/\Gamma$$

where $\Gamma \subset SL_2(\mathbb{R})$ is a discrete subgroup (with torsion-free image in $PSL_2(\mathbb{R})$).

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- Any closed geodesic c in M corresponds to a semi-simple $\gamma \in \Gamma$, i.e. $c = c_\gamma$.

- It has *length*

$$\ell(c_\gamma) = (1/n_\gamma) \cdot \log t_\gamma$$

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$$\ell(c_\gamma) = (1/n_\gamma) \cdot \log t_\gamma$$

where t_γ is eigenvalue of $\pm\gamma$ which is > 1,

n_γ is an integer $\geqslant 1$.

NOTE: $\pm\gamma$ is conjugate to $\begin{pmatrix} t_\gamma & 0 \\ 0 & t_\gamma^{-1} \end{pmatrix}$.

Geometric perspective

If $M_i = \mathbb{H}/\Gamma_i \quad (i = 1, 2)$ are length-commensurable **then**:

- for *any* nontrivial semi-simple $\gamma_1 \in \Gamma_1$ *there exists* a nontrivial semi-simple $\gamma_2 \in \Gamma_2$ such that

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So,

$$t_{\gamma_1}^{n_1} = t_{\gamma_2}^{n_2}$$

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This means that

$$\chi_1(\gamma_1) = \chi_2(\gamma_2) \neq 1$$

where χ_i is the character of the maximal $\mathbb{R}$-torus $T_i \subset \mathrm{SL}_2$ corresponding to $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \mapsto t^{n_i}$.

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It follows that

Γ_1 and Γ_2 are weakly commensurable.

Outline

Philosophy: *An arithmetic group is a group that "looks like" $SL_n(\mathbb{Z})$.*

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More generally: For a number field K and a set S of places of K, containing all archimedean ones, $\mathcal{O}(S)$ denotes ring of S-integers.

E.g.: If $K = \mathbb{Q}$ and $S = \{\infty, 2\}$ then $\mathcal{O}(S) = \mathbb{Z}[1/2]$.

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More generally: For a number field K and a set S of places of K, containing all archimedean ones, $\mathcal{O}(S)$ denotes ring of S-integers.

Given an algebraic K-group $G \subset \mathrm{GL}_n$, set $G(\mathcal{O}(S)) = G \cap GL_n(\mathcal{O}(S))$; subgroups of $G(F)$ (F/K) commensurable with $G(\mathcal{O}(S))$ are (K, S)-arithmetic.

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We can consider rational quadratic forms $\mathbb{R}$-equivalent to f:

$$f_1 = x^2 + y^2 - 3z^2 \quad \text{or} \quad f_2 = x^2 + 2y^2 - 7z^2.$$

Then $\mathrm{SO}_3(f_i) \simeq \mathrm{SO}_3(f)$ over $\mathbb{R}$, and

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One can also consider $K = \mathbb{Q}(\sqrt{2}) \subset \mathbb{R}$ and $f_3 = x^2 + y^2 - \sqrt{2}z^2$. Then

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One can further replace integers by S-integers, etc.

Definition of arithmeticity

Definition. Let G be an absolutely almost simple algebraic group over a field F, $\mathrm{char}\, F = 0$, and $\pi \colon G \to \overline{G}$ be isogeny onto adjoint group.

1. a number field K with a *fixed* embedding $K \hookrightarrow F$;
2. a finite set $S \subset V^K$ containing V^K_∞;
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Then subgroups $\Gamma \subset G(F)$ such that $\pi(\Gamma)$ is commensurable with $\mathcal{G}(\mathcal{O}_K(S))$ are called $(\mathcal{G}, K, S)$-arithmetic.

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We do NOT fix an F-isomorphism ${}_F\mathcal{G} \simeq \overline{G}$ in n° 3; by varying it we obtain a class of groups invariant under F-automorphisms.

Proposition. *Let G_1 and G_2 be connected absolutely almost simple algebraic groups defined over a field F, $(\mathrm{char}\, F = 0)$, and let $\Gamma_i \subset G_i(F)$ be a Zariski- dense $(\mathcal{G}_i, K_i, S_i)$-arithmetic group $(i = 1, 2)$.*

Then Γ_1 and Γ_2 are commensurable up to an F-isomorphism between $\overline{G}_1$ and $\overline{G}_2$ if and only if

- $K_1 = K_2 =: K$;
- $S_1 = S_2$;
- $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *are* K-*isomorphic.*

In above example, Γ_1, Γ_2 and Γ_3 are *pairwise* noncommensurable.

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- Γ_1 and Γ_2 are NOT commensurable b/c the corresponding $\mathbb{Q}$-forms $\mathcal{G}_1 = \mathrm{SO}_3(f_1)$ and $\mathcal{G}_2 = \mathrm{SO}_3(f_2)$ are NOT isomorphic over $\mathbb{Q}$.

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- Γ_3 is NOT commensurable to either Γ_1 or Γ_2 b/c they have different fields of definition:

$$\mathbb{Q}(\sqrt{2}) \text{ for } \Gamma_3, \quad \text{and } \mathbb{Q} \text{ for } \Gamma_1 \text{ and } \Gamma_2.$$

Outline

Results of Prasad-R. and follow-up results Garibaldi, Garibaldi-R. provide a (virtually) complete analysis of weak commensurability for arithmetic groups.

In particular:

- we know when weak commensurability $\Rightarrow$ commensurability (answer depends on Lie type of algebraic group)
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If there exist finitely generated Zariski-dense subgroups $\Gamma_i \subset G_i(F)$ $(i = 1, 2)$ that are weakly commensurable then

either G_1 and G_2 have the same Killing-Cartan type, or one of them is of type B_n and the other is of type C_n $(n \geqslant 3)$.

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NOTE: groups of types B_n and C_n can indeed contain Zariski-dense weakly commensurable subgroups.

Theorem 2. *Let* $\Gamma_i \subset G_i(F)$ *be a Zariski-dense* $(\mathcal{G}_i, K_i, S_i)$*-arithmetic subgroup for* $i = 1, 2$.

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The forms $\mathcal{G}_1$ and $\mathcal{G}_2$ may NOT be K-isomorphic in general,

but we have the following.

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Theorem 3. *Let* G_1 *and* G_2 *be of the same type different from* A_n, D_{2n+1} *with* $n > 1$, *and* E_6, *and let* $\Gamma_i \subset G_i(F)$ *be a Zariski-dense* $(\mathcal{G}_i, K, S)$*-arithmetic subgroup.*

If Γ_1 *and* Γ_2 *are weakly commensurable then* $\mathcal{G}_1 \simeq \mathcal{G}_2$ *over* K, *and hence* Γ_1 *and* Γ_2 *are commensurable (up to an F-isomorphism between* $\overline{G}_1$ *and* $\overline{G}_2$*).*

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For types A_n, D_{2n+1} $(n > 1)$ and E_6 we have counterexamples.

Theorem 4. *Let $\Gamma_1 \subset G_1(F)$ be a Zariski-dense (K,S)-arithmetic subgroup.*

Then the set of Zariski-dense (K,S)-arithmetic subgroups $\Gamma_2 \subset G_2(F)$ that are weakly commensurable to Γ_1, is a union of finitely many commensurability classes.

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Theorem 5. *Let $\Gamma_i \subset G_i(F)$ be a Zariski-dense $(\mathcal{G}_i, K, S)$-arithmetic subgroup for $i = 1, 2$.*

If Γ_1 and Γ_2 are weakly commensurable then $\mathrm{rk}_K\, \mathcal{G}_1 = \mathrm{rk}_K\, \mathcal{G}_2$; in particular, if $\mathcal{G}_1$ is K-isotropic then so is $\mathcal{G}_2$.

Outline

Notations

- G a connected absolutely (almost) simple algebraic group $/\mathbb{R}$; $\mathcal{G} = G(\mathbb{R})$
- $\mathcal{K}$ a maximal compact subgroup of $\mathcal{G}$; $\mathfrak{X} = \mathcal{K}\backslash\mathcal{G}$ associated symmetric space, $\operatorname{rk}\mathfrak{X} = \operatorname{rk}_{\mathbb{R}} G$
- Γ a discrete torsion-free subgroup of $\mathcal{G}$, $\mathfrak{X}_\Gamma = \mathfrak{X}/\Gamma$
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Given G_1, G_2, $\Gamma_i \subset \mathcal{G}_i := G_i(\mathbb{R})$ etc. as above, we will denote corresponding *locally symmetric spaces* by $\mathfrak{X}_{\Gamma_i}$.

Fact. *Assume that* $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are of finite volume. If* $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are length-commensurable then* (under minor technical assumptions) Γ_1 *and* Γ_2 *are weakly commensurable.*

- in rank one case - on the result of Gel′fond and Schneider (1934): *if* α *and* β *are algebraic numbers* $\neq 0, 1$, *then* $\dfrac{\log \alpha}{\log \beta}$ *is either rational or transcendental.*
- in higher rank case - on the following

Conjecture (Shanuel) *If* $z_1, \ldots, z_n \in \mathbb{C}$ *are linearly independent over* $\mathbb{Q}$, *then the transcendence degree of field generated by*

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Outline

Theorem 6. *Let* $\mathfrak{X}_{\Gamma_1}$ *be an arithmetically defined locally symmetric space.*

• *The set of arithmetically defined locally symmetric spaces* $\mathfrak{X}_{\Gamma_2}$ *that are length-commensurable to* $\mathfrak{X}_{\Gamma_1}$, *is a union of finitely many commensurability classes.*

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Theorem 7. *Let* $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *be locally symmetric spaces of finite volume, and* **assume** *that one of the spaces is arithmetically defined.*

If $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *are length-commensurable* **then** *compactness of one of the spaces implies compactness of the other.*

Theorem 8. *Let* $\mathfrak{X}_{\Gamma_1}$ *and* $\mathfrak{X}_{\Gamma_2}$ *be isospectral compact locally symmetric spaces.*

If $\mathfrak{X}_{\Gamma_1}$ *is arithmetically defined* **then** *so is* $\mathfrak{X}_{\Gamma_2}$.

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Theorem 9. *Let $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ be isospectral compact locally symmetric spaces, and assume that at least one of the spaces is arithmetically defined.*

Then $G_1 = G_2 =: G$.

Moreover, unless G is of type A_n, D_{2n+1} $(n > 1)$ or E_6, spaces $\mathfrak{X}_{\Gamma_1}$ and $\mathfrak{X}_{\Gamma_2}$ are commensurable.