$^2E_6(2)$ and the Baby Monster a match made in heaven

G. Stroth

Martin Luther Universität Halle Wittenberg

March 11, 2012

${}^2E_6(q)$

Let C_q be the centralizer of a 2-central involution in ${}^2E_6(q)$, $q = 2^n$. If G is a finite simple group containing an involution z, such that $C_G(z) \cong C_q$, then G is isomorphic to ${}^2E_6(q)$.

${}^2E_6(2)$

Let $H_1 = {}^2E_6(2)$. Then there is a group H_2 such that

$$H_2/Z(H_2) \cong H_1, |Z(H_2)| = 2$$

but there is no subgroup isomorphic to H_1 in H_2.

Furthermore H_1 has an outer automorphism t of order 2 and this can be extended to H_2.

We denote by H^* the semidirect product of H_2 by $\langle t \rangle$. This group is isomorphic to the centralizer of an involution in BM.

The Baby Monster

Let d_1 be a fixed element in the conjugacy class D^* of elements of BM. Denote by $H^* = C_{BM}(d_1)$. Let G be a finite simple group containing an involution d whose centralizer in G is isomorphic to H^*. Then

(i) $|G| = 2^{41} \cdot 2^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$;

(ii) *Let C be the conjugacy class of d in G, then C is a class of $\{3, 4\}$-transpositions and*

(iii) *Denote by $H = C_G(d)$, then H has five orbits on C.*

The Baby Monster

There are four classes of involution, with representatives

$$z, d, t \text{ and } e.$$

$$C_{BM}(t) \cong (\langle t, d\rangle \times F_4(2)) : 2$$

$$C_{BM}(z) \cong 2^{1+22}Co_2$$

$$C_{BM}(e)/O_2(C_{BM}(e)) \cong O^+(8,2)$$

$$Z(O_2(C_{BM}(e)))/\langle e\rangle \text{ is a natural module}$$

We have that $C_{BM}(e)$ is contained in a 2-local subgroup M such that $M/O_2(M) \cong Sp(8,2)$, there is a natural submodule $V \leq Z(O_2(M))$, $O_2(M)/V$ is extraspecial and $O_2(M)/Z(O_2(M))$ is the spin module.

The Baby Monster

Thompsons order formula:

Let $i_1, \ldots, i_m$, $m \geq 2$ be representatives of the conjugacy classes of involutions of G.

Let $a(i_k)$ be the number of ordered pairs (i,j) with $i \sim i_1$, $j \sim i_2$ and $i_k = (ij)^n$ for some n. Then

$$|G| = |C_G(i_2)|a(i_1) + |C_G(i_1)|a(i_2) + \sum_{k=3}^m (|C_G(i_1)||C_G(i_2)|/|C_G(i_k)|)a(i_k)$$

The Baby Monster

Choose $i_1 = z$ and $i_2 = t$

Then one can show

$$a(d) = 0 = a(e)$$

$$a(t) \leq 3^4 \cdot 5^2 \cdot 7^2 \cdot 13^2 \cdot 17^2 \cdot 3971$$

$$a(z) \leq 2^{15} \cdot 3^4 \cdot 5^2 \cdot 11 \cdot 13 \cdot 23^2 \cdot 713$$

Hence

The Baby Monster

Choose $i_1 = z$ and $i_2 = t$

Then one can show

$$a(d) = 0 = a(e)$$

$$a(t) \leq 3^4 \cdot 5^2 \cdot 7^2 \cdot 13^2 \cdot 17^2 \cdot 3971$$

$$a(z) \leq 2^{15} \cdot 3^4 \cdot 5^2 \cdot 11 \cdot 13 \cdot 23^2 \cdot 713$$

Hence

$|G| = 2^{41} \cdot 3^{10} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot r$ with $r \leq 176869$.

The Baby Monster

Choose $i_1 = z$ and $i_2 = t$

Then one can show

$$a(d) = 0 = a(e)$$
$$a(t) \leq 3^4 \cdot 5^2 \cdot 7^2 \cdot 13^2 \cdot 17^2 \cdot 3971$$
$$a(z) \leq 2^{15} \cdot 3^4 \cdot 5^2 \cdot 11 \cdot 13 \cdot 23^2 \cdot 713$$

Hence

$|G| = 2^{41} \cdot 3^{10} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot r$ with $r \leq 176869$.

$|G| = 2^{41} \cdot 3^{10} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 39339$

The Baby Monster

Let C be the conjugacy class of d. Then we see that there are elements in H' different to d, which are conjugate to d and there a some in $H \setminus H'$, which are conjugate to d.

Let C be the conjugacy class of d. Then we see that there are elements in H' different to d, which are conjugate to d and there a some in $H \setminus H'$, which are conjugate to d.

This yields H-orbits of length 1,

The Baby Monster

Let C be the conjugacy class of d. Then we see that there are elements in H' different to d, which are conjugate to d and there a some in $H \setminus H'$, which are conjugate to d.

This yields H-orbits of length 1, $3^3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 19$

The Baby Monster

Let C be the conjugacy class of d. Then we see that there are elements in H' different to d, which are conjugate to d and there a some in $H \setminus H'$, which are conjugate to d.

This yields H-orbits of length 1, $3^3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 19$ and $2^{12} \cdot 3^3 \cdot 11 \cdot 19$ on C.

The Baby Monster

Let C be the conjugacy class of d. Then we see that there are elements in H' different to d, which are conjugate to d and there a some in $H \setminus H'$, which are conjugate to d.

This yields H-orbits of length 1, $3^3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 19$ and $2^{12} \cdot 3^3 \cdot 11 \cdot 19$ on C.

Furthermore as $d \in O_2(C_G(z))$, there are conjugates $\tilde{d}$ with $(d\tilde{d})^2 = z$. This yields an orbit of length $2^8 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$.

The Baby Monster

Let C be the conjugacy class of d. Then we see that there are elements in H' different to d, which are conjugate to d and there a some in $H \setminus H'$, which are conjugate to d.

This yields H-orbits of length 1, $3^3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 19$ and $2^{12} \cdot 3^3 \cdot 11 \cdot 19$ on C.

Furthermore as $d \in O_2(C_G(z))$, there are conjugates $\tilde{d}$ with $(d\tilde{d})^2 = z$. This yields an orbit of length $2^8 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$.

The Baby Monster

We can find two conjugates of d, which generate a Σ_3 and centralize in $C_H(z)$ a double cover of $U_6(2)$.

This then implies that they centralize in H a subgroup $M(22)$. So we get a further orbit of length $2^{20} \cdot 7 \cdot 17 \cdot 19$.

The Baby Monster

We can find two conjugates of d, which generate a Σ_3 and centralize in $C_H(z)$ a double cover of $U_6(2)$.

This then implies that they centralize in H a subgroup $M(22)$. So we get a further orbit of length $2^{20} \cdot 7 \cdot 17 \cdot 19$.

These orbit length add up to $|C|$. So C is a class of $\{3, 4\}$-transpositions.

Remarks

The centralizer of a 2-central involution in ${}^2E_6(2)$ contains a subgroup $U_6(2)$ and a subgroup $2U_6(2)$. The first one centralizes an element of order three inverted by two conjugates of the 2-central involution, the second one embeds into $M(22)$.

Remarks

The centralizer of a 2-central involution in ${}^2E_6(2)$ contains a subgroup $U_6(2)$ and a subgroup $2U_6(2)$. The first one centralizes an element of order three inverted by two conjugates of the 2-central involution, the second one embeds into $M(22)$.

The centralizer of d in BM contains a subgroup $M(22)$ and a subgroup $2M(22)$. The first one centralizes an element of order three, which is inverted by two conjugates of d, while the second one embeds into $M(23)$.

Remarks

The centralizer of a 2-central involution in ${}^2E_6(2)$ contains a subgroup $U_6(2)$ and a subgroup $2U_6(2)$. The first one centralizes an element of order three inverted by two conjugates of the 2-central involution, the second one embeds into $M(22)$.

The centralizer of d in BM contains a subgroup $M(22)$ and a subgroup $2M(22)$. The first one centralizes an element of order three, which is inverted by two conjugates of d, while the second one embeds into $M(23)$.

A geometry for BM

We come back to the $\{3, 4\}$-transpositions d in BM.

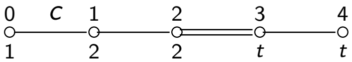
Consider the graph which consists of these involutions and adjacency is given if two such involutions commute and the product is a 2-central involution in BM.

Then the adjacent involutions to a given one correspond to the root involutions of ${}^2E_6(2)$, i.e. locally is the graph the F_4–building.

As said before there is a 2-local subgroup M such that $M/O_2(M) \cong Sp_8(2)$. In $Z(O_2(M))$ there are exactly 120 conjugates of d, which form a maximal clique in the graph.

A geometry for BM

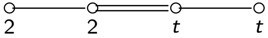
If we consider a geometry whose objects are the intersections of these cliques and incidence is given by inclusion, we get a Buekenhout geometry with diagram



Where $t = 4$.

A geometry for BM

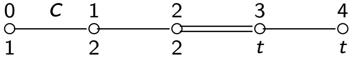
According to Tits we have three buildings with diagram



with $t = 1, 2$ and 4 and flag transitive automorphism group

$$\Omega_8^+(2) : S_3, F_4(2), {}^2E_6(2) : S_3.$$

Hence we might have three geometries of type



We will call them $cF_4(t)$-geometries

$cF_4(4)$-geometries

A. Ivanov, 1992 showed

If Γ is a graph which is locally isomorphic to the Baby monster graph and the automorphism group of Γ induces at least ${}^2E_6(2)$ on the neighbors of any vertex, then Γ is the Babymonster graph.

More generally A. Ivanov, D. Pasechnik and S. Shpectorov 2001 showed that a $cF_4(4)$ geometry is the Baby monster geometry provided the following two conditions hold

(a) any two elments of type 1 are incident with at most one of type 2

(b) Three elements of type 1 are pairwise incident with an element of type 2 if and only if they are incident with an element of type 5

$cF_4(1)$-geometries

Let $G = M(22) : 2$ and i be a $2D$-involution. Then $C_{M(22)}(i) \cong \Omega_8^+(2) : S_3$.

Consider the graph for G whose vertices are these involutions and two are adjacent if they commute. Maximal cliques are of order 36 and generate an elementary abelian group of order 2^7 on which $Sp_6(2)$ acts.

Now consider the geometry as before, whose objects are the intersections of those cliques. Then we have a $cF_4(1)$-geometry for $M(22) : 2$.

One also can construct this geometry as a subgeometry of the Baby monster geometry.

$cF_4(1)$-geometries

A. Ivanov and C. Wiedorn 2002 showed that a $cF_4(1)$-geometry which satisfies (a) and (b) before is the geometry for $M(22)$ or its triple cover $3M(22)$.

$cF_4(2)$-geometries

We are left with $cF_4(2)$-geometries.

First some examples:

We have seen in the Baby monster that there is an outer involution t in $G = {}^2E_6(2) : 2$, which centralizes in ${}^2E_6(2)$ a subgroup $H \cong F_4(2)$. The same applies for the graph automorphism t of $E_6(2)$. There is a root involution $r \in H$ such that $tr \sim t$ in G.

We consider the graph whose vertices are the conjugates of t and two are connected, if they commute and differ by a root involution. Then maximal cliques in this graph have order 64. As before we now construct a $cF_4(2)$-geometry, where the residue of a point is just the $F_4(2)$–building.

${}^2E_6(2)$, $E_6(2)$

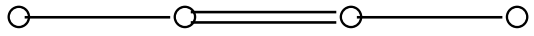
We now have two $cF_4(2)$-geometries, which have flag transitve acting groups

$$E_6(2), E_6(2):2, {}^2E_6(2) \text{ and } {}^2E_6(2):2.$$

The latter one has a covering which belongs to $3 \cdot {}^2E_6(2)$ and has also the group $3 \cdot {}^2E_6(2):2$ acting.

A representation group for $F_4(2)$

There is a further example. The group $F_4(2)$ has an irreducible module of dimension 26 over the field with two elements. In fact we can see the building



in this module. The elements of type one correspond to some 1-spaces, type 2 correspond to 2-space, type 3 to 3-space and type 4 to 6-spaces, which all consist just of vectors of the first type.

Now take the vectors of V as new objects of type 0 and the objects of type i, $i = 1, 2, 3, 4$ are just the cosets of the objects of the old types i in V. This then gives a $cF_4(2)$-geometry on which $VF_4(2)$ acts flag transitively.

The classification

The classification of all $cF_4(2)$-geometries, which satisfy (a), (b) before has been achieved by C. Wiedorn 2007 and G. Stroth 2012.

The main idea due to Corinna Wiedorn was not to work with the geometry but to determine the amalgam of the maximal parabolics.

She wanted to show that the amalgams are exactly those as in the examples above. Then using a result due to A. Ivanov, A. Pasini 2003 one is done.

The amalgams

First of all she showed that the kernel K_0 of the action of an point stabilizer G_0 on the residue of this point is of order at most two.

She showed that for $K_0 = 1$ one has the $2^{26}F_4(2)$ example and nothing else and in the other case one has $G_0 = K_0 \times F_4(2)$.

amalgams

Corinna Wiedorn was able to get an upper bound on the number of possible amalgams which is eight. This was done by checking how many isomorphism types for certain group extensions exist and so she achieved this bound. Unfortunately she was not able to finish the work.

Investigating what she had done and proving that some isomorphism types do not fit together to build an amalgam, I was able to prove that there are exactly four such amalgams. Hence we have

A flag transitive $cF_4(2)$-geometry satisfying (a) and (b) is one belonging to $2^{26}F_4(2)$, $3 \cdot {}^2E_6(2):2$, ${}^2E_6(2):2$ or $E_6(2):2$.

As the minimal circuit diagram of a $cF_4(t)$-geometry is of type $\tilde{F}_4$ it would not wonder if there are infinite universal coverings in general.

Also $F_4(2)$ is a representation group for the F_4-building. Hence $F_4(2) \times F_4(2)$ acts flag transitively on a $cF_4(2)$-geometry. This example does not satisfy (b).

Groups with large p-subgroups

Let $G = G(q)$, $q = p^f$, be a group of Lie type and R be a root subgroup in G. Set $C = N_G(R)$ and denote by Q the p-radical of C.

Then in almost all cases $C = N_G(Q)$, $Q = O_p(C)$, $C_C(Q) \leq Q$, $R = Z(Q)$ and $N_G(R_1) \leq C$ for all $1 \neq R_1 \leq R$.

Groups with large p-subgroups

In order to classify the groups of Lie type there is a generalization of this observation:

Let G be a group and $Q \leq G$ be a p-group for some prime p. We call Q a large subgroup of G if

- $Q = O_p(N_G(Q))$, $C_G(Q) \leq Q$.
- If U is a non-trivial subgroup of $Z(Q)$, then $N_G(U) \leq N_G(Q)$.

Groups with large p-subgroups

There is a project due to U. Meierfrankenfeld, B. Stellmacher, G Stroth to identify the groups G, with large p-subgroup Q and under a $\mathcal{K}_p$-group assumption

This will be done under a further hypothesis:

Let S be a Sylow p-subgroup of $N_G(Q)$. There is some $1 \neq E \trianglelefteq S$ such that $N_G(E) \not\leq N_G(Q)$.

The Structure Theorem, The H-Structure Theorem

In a first step infomation about the structure of $N_G(E)$ is obtained. (with the additional assumption that $F^*(N_G(F)) = O_p(N_G(F))$ for all $1 \neq F \leq S$.)

Afterwards some infomation about $N_G(Q)$ will be used to find a minimal parabolic P of $N_G(Q)$, which is not contained in $N_G(E)$.

Hence $H = \langle P, N_G(E) \rangle$ is a subgroup of G with $O_p(H) = 1$.

In the generic case one can show that H is an automorphism group of a group of Lie type in characteristic p.

The group $N_G(Q)$

One would expect that $H = G$.

In a first step one should be able to prove that $N_G(Q) \leq H$.

Assume that the Lie rank of $F^*(H)$ is at least three and $N_H(Q)$ is nonsolvable, then G. Pientka and A. Seidel could prove that $N_G(Q) \leq H$.

From now on let p be odd. If $N_H(Q)$ is solvable then

$$H^* = F^*(H) \cong \Omega_n(3), n \leq 8.$$

$^2E_6(2)$

If $H^* \cong \Omega_7(3)$ we have

$$N_{H^*}(Q) \approx 3_+^{1+6}.(\mathrm{SL}_2(3) \times \Omega_3(3)).2.$$

$(\mathrm{SL}_2(3) \times \Omega_3(3)).2$ is not a maximal subgroup of the automorphism group of Q. This is

$$3_+^{1+6}(\mathrm{Sp}_2(3) \wr \mathrm{Sym}(3))$$

In fact there is a series of groups

$$\Omega_7(3) < M(22) < {}^2E_6(2)$$

corresponding to different overgroups of $N_{H^*}(Q)$.

BM

If $H^* \cong \Omega_8^+(3)$ we have

$$N_{H^*}(Q)/Q \approx (\mathrm{SL}_2(3) * \mathrm{SL}_2(3) * \mathrm{SL}_2(3)) : 2 \approx 2_-^{1+6}.3^3.2.$$

Again this is not maximal. The maximal group is

$$N_{H^*}(Q) \approx 3_+^{1+8}.2_-^{1+6}.U_4(2) : 2$$

There is a series

$$\Omega_8^+(3) < M(23) < BM$$

corresponding to different overgroups of $N_{H^*}(Q)$.

Almost groups of Lie type

Assume that p is a prime, G is a group, $H = N_G(F^*(H))$ contains a Sylow p-subgroup of G and $F^*(H)$ is a simple group of Lie type in characteristic p and rank at least two but that $F^*(H) \not\cong \mathrm{PSL}_3(p^a)$ when p is odd.
Suppose that a large subgroup Q of G is contained in H and $C_H(z)$ is soluble for some p-central element z of G. Then one of the following holds (Chr.Parker, G. Stroth):

1. $N_G(Q) = N_H(Q)$;
2. $p = 2$ *and* $F^*(G) \cong M_{11}, M_{23}$, $\mathrm{G}_2(3)$ *or* $\mathrm{P}\Omega_8^+(3)$; *or*
3. $p = 3$ *and* $F^*(G) \cong U_6(2)$, $F_4(2)$, ${}^2E_6(2)$, McL, Co_2, $M(22)$, $M(23)$ *or* BM.

Almost groups of Lie type

Due to the work of A. Seidel, G. Pientka, Chr. Parker and myself we have

Let $F^*(H) \leq G$ be a group of Lie type in characteristic p and of rank at least two not isomorphic to $L_3(p^a)$ for p odd. Suppose that there is a large subgroup Q of G such that $N_G(Q) = N_H(Q)$. Then one of the following holds

a) *If S is a Sylow p-subgroup of H then $H = \langle N_G(U) \mid 1 \neq U \trianglelefteq S \rangle$; or*

b) *$p = 2$ and $H = \Sigma_8$ and $G = A_{10}$ or $H = U_4(2) : 2$ and $G = L_4(3)$.*

Strongly embedded

Let $F^*(H) \leq G$ be a group of Lie type in characteristic p and of rank at least two if $p = 2$ and rank at least three if p is odd. Let S be a Sylow p-subgroup of H and $H = \langle N_G(U) \mid 1 \neq U \trianglelefteq S \rangle$.

(Chr. Parker, R. Salarian, G. Stroth) If G is of local characteristic p and G is a K_p-group, then $G = H$.

Work in progress M. Grimm: If $p = 2$ and G is of parabolic characteristic 2 and a K_2-group, then either $G = H$ or $F^*(H) = L_4(2)$ and $F^*(G) = A_9$.

aim

Let G be a K_2-group of local (parabolic) characteristic 2 with a large 2-subgroup Q, containing a subgroup H, such that $F^*(H)$ is a group of Lie type in characteristic 2 and rank at least 2 and $|G:H|$ is odd. Then either $G = H$ or

$G = M_{11}$, M_{23} or $G_2(3)$. ($F^*(G) \cong \Omega_8^+(3)$, A_9, A_{10}, or $L_4(3)$)