Parapolar spaces and large sporadic groups

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A parapolar space is a point line geometry with main axiom: If P and Q are points of distance two, then either $|P^\perp \cap Q^\perp| = 1$ or $P^\perp \cap Q^\perp$ is (with the lines contained in it) a polar space. In the first case P, Q is called a *special pair* and the unique point in $P^\perp \cap Q^\perp$ will be called $P(P,Q)$.

Example: Root group geometries in exceptional buildings of type F_4, E_6, E_7 or E_8

The connection of parapolar spaces with sporadic groups is given by large extraspecial 2-subgroups.

Definition A 2-group Q is *extraspecial*, if $Z(Q) = \langle a \rangle \simeq \mathbb{Z}_2$ and $\tilde{Q} = Q/\langle a \rangle$ is elementary abelian. The extraspecial 2-group Q is *large* inside some finite simple group G, if $Q = O_2(C(a))$ and $C(Q) \leq Q$.

In this situation we call $Q = Q_a$, $M = M_a = C(a)$, $\bar{M} = M/Q$ and ~~$\tilde{Q}$~~, if $b = a^g$, then $Q_b = Q^g$, $M_b = M^g$.
$\tilde{Q}$ carries the structure of a nondegenerate orthogonal space over F_2 (with quadratic form given by square map). So $|\tilde{Q}| = 2^{2n}$ for some n.

The classification of finite simple groups with large extraspecial 2-subgroup has been an important part of the classification of finite simple groups, see the new book by M. Aschbacher, R. Lyons, St. Smith and R. Solomon. More than half of the sporadic groups possess such a large extraspecial 2-subgroup.

In my talk I will from now on assume

__Hypothesis (H)__. Q is a large extraspecial 2-subgroup of the finite simple group G and:

(1) $n \geq 4$ for $|\tilde{Q}| = 2^{2n}$.

(2) There exist $a^g \in M - Q$ and $a^h \in Q - \langle a \rangle$.

(3) $\bar{M}$ acts irreducibly on $\tilde{Q}$.

(4) There is no $t \in M - Q$ such that $[\tilde{Q}, t]$ is a singular line.

The splitting of the cases ① – ③ reflects a historical subdivision. I.e these cases have been dealt with before one started to consider the large extraspecial problem in general. ④ is a special and easier case. Namely such a t is a Siegel-transformation on $\tilde{Q}$ and thus by ③ $\bar{H}$ is an irreducible subgroup of $O(\tilde{Q})$ generated by Siegel-transformations and whence known.

We have

<u>Proposition 1</u> Suppose G satisfies (H). Let $\mathcal{P} = a^G$ and $\mathcal{L} = \{(a, b, ab) \mid a \in \mathcal{P},\ b \in \mathcal{P} \text{ and } b \in Q_a\}$. Then $(\mathcal{P}, \mathcal{L}, \in)$ is a parapolar space.

Notice that the definition of lines is symmetric, i.e $a \in Q_b$ if $b \in Q_a$.

<u>Proposition 2.</u> Let $a, b \in \mathcal{P}$ of distance two such that $a^\perp \cap b^\perp$ is a nondegenerate polar space. Let $S(a,b) = \langle a, a^\perp \cap b^\perp, b \rangle$, $R = \langle Q_a, Q_b \rangle$ and $N = (Q_a \cap C(b))(Q_b \cap C(a))$. Then

(1) $N = O_2(R)$ and $R/N \simeq \Omega^{\pm}(2m,2)$ or $Sp(2m,2)$ acting naturally on $S(a,b)$.

(2) The structure of N and the action of R on N is completely determined.

(If $R/N \simeq \Omega^{\pm}(2m,2)$ then $N/S(a,b)$ is the direct sum of spin modules for R/N.)

Call a parapolar space nondegenerate, if there exists a pair a, b of points of distance two with $|a^{\perp} \cap b^{\perp}| \geq 2$ and if for each such pair $a^{\perp} \cap b^{\perp}$ is a nondegenerate polar space. Then we have:

Theorem 1 Suppose the parapolar space of Prop. 1 is nondegenerate. Then one of the following holds.

(a) If $a, b \in \mathcal{P}$ is a special pair, then $P(a,b) = (ab)^2 = [a,b]$. Then $\mathcal{P}$ is a class of $\{3,4\}^+$-transpositions of G and G is isomorphic to $\Omega^{\pm}(2m,2)$, $m \geq 4$, $E_6(2)$, $E_7(2)$, $E_8(2)$ or ${}^2E_6(2)$

(b) There exists a special pair a, b of commuting involutions and $n = 6, 11$ or 12. $(|\tilde{Q}| = 2^{2n}.)$

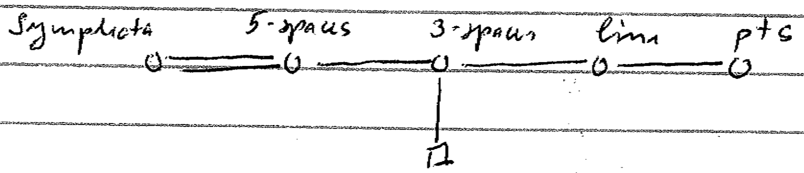
Theorem 2. Suppose case (B) of theorem 1 holds. Then

(1) If $n = 12$, then $G \simeq M$ – the Monster

(2) If $n = 11$, then $G \simeq BM$ – the Baby-monster.

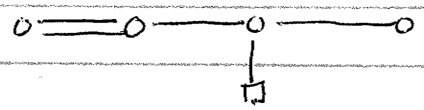
Sketch of proof. If $n = 12$, then $R/N \simeq \Omega^+(10,2)$, $S(a,b)$ is the natural module for R/N and $N/S(a,b)$ is the spin module for R/N. Further $R = N(S(a,b))$. In particular $C(a) \cap C(b) \subseteq R$. From the structure of R one sees that $C_{\bar{M}}(\bar{b}) \simeq 2^{1+8}\Omega^+(8,2)$, where $M = C(a)$. Hence $\bar{M} \simeq Co_1$, since we assume inductively that the case (H)(4) is already treated. It is easy to see that $\hat{Q}$ is the Leech-lattice mod 2 as $\bar{M}$-module.

Let Γ be the geometry with vertices the "symplecta" $S(a,b)$ and those singular subspaces of $(\mathcal{P}, \mathcal{L}, \in)$, the stabilizers of which are maximal subgroups. Then Γ is a diagram geometry with diagram.



(Here □ stands for a non existing vertex)

This is easy to see, since the singular subspaces contained in $S(a,b)$ form a D_5-geometry. Now the stabilizer of one class of singular 5-spaces in $S(a,b)$ is not maximal, while the stabilizers of the other singular subspaces of $S(a,b)$ are maximal.

From the action of $\bar{M}$ on $\tilde{Q}$ one sees that the residue of a point is the Ronan-Smith geometry for Co_1, i.e. has diagram  .

This shows that Γ has the above diagram. Now Ivanov shows that this geometry is simply connected (under some additional conditions, which are easily verified, since we know R and M.) Hence $G \cong M$.

In case $n = 11$ one can show that the geometry of ~~the~~ singular subspaces of the parapolar space is the Petersen geometry of BM. Since this geometry is by Ivanov also simply connected, we obtain $G \cong BM$.

In the case when the parapolar space of Prop. 1 is degenerate one also obtains few cases which correspond to the sporadic Thompson group, the Harada-Norton group or to J_4 (the 4-th Janko group.).