3-Transpositions and Moufang Quadrangles

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Bielefeld, March 14, 2012

Today—roughly—is the 41st anniversary of the publication of Bernd Fischer's famous classification of groups generated by 3-transpositions. I would like to contribute to this celebration in honor of Professor Fischer by recalling a few details of the classification in the first part of this talk.

There are uncanny parallels between Fischer's classification and the classification of Moufang quadrangles. In the second half of this talk, I want to try describe some of these parallels and say something about some recent work with Bernhard Mühlherr and Holger Petersson.

So: let G be a finite group generated by a set D of 3-transpositions. This means that D is set of involutions (i.e. a set of elements of order 2) closed under conjugation and $|de| \leq 3$ for all $d, e \in D$.

We let $\Gamma = [D]$ denote the commuting graph on D and we let $\bar{\Gamma} = (D)$ denote its complement. If $d \in D$, then D_d is its set of neighbors in the graph Γ and A_d its set of neighbors in the graph $\bar{\Gamma}$.

If $e \in A_d$, then $|de| = 3$ and hence e is conjugate to d in $\langle d, e \rangle$. It follows that G acts transitively on D (by conjugation) if and only if $\bar{\Gamma}$ is connected.

A crucial observation:

- D_d is also a set of 3-involutions.

Consequently, the study of 3-transpositions is inherently inductive.

D is called *reduced* if no two elements of D have the same sets of neighbors, neither in Γ nor in $\bar{\Gamma}$ (and also $|D| > 1$).

If D is the set of transpositions in S_3 or S_4, for example, then D is not reduced.

The classification says that if

$$(*) \qquad [D] \text{ and } (D) \text{ are connected and } D \text{ is reduced,}$$

then $\Gamma = [D]$ comes from a symmetric, orthogonal, symplectic or unitary group or from one of the three sporadic groups Fi_{22}, Fi_{23} or Fi_{24}.

The proof of the classification rests on the following:

Fundamental Lemma: $(*)$ implies that $\langle d, D_d \rangle$ acts transitively on both D_d and A_d.

Here is an elementary consequence of the Fundamental Lemma:

Proposition: If for each $d \in D$, (D_d) is connected and D_d is reduced, then $[D_d]$ is connected.

Proof: Since no two elements of D_d have the same set of neighbors in $[D_d]$, we can choose an edge $\{u, v\}$ of $[D_d]$. Since no two elements of D_d have the same set of neighbors in (D_d), there is a $w \in D_d$ adjacent to v but not u. Then $\{d, v\}$ is an edge of the subgraph $[D_u \cap D_w]$. If, on the other hand, x, y are elements of D_d contained in different components of $[D_d]$, then the subgraph $[D_x \cap D_y]$ contains no edges. It is easy to see, however, that we can apply the Fundamental Lemma, which implies that (u, w) is in the same G-orbit as (x, y). QED

The classification of 3-transpositions has three parts. Part I is, essentially, the proof of the Fundamental Lemma. In part II, the classification is carried out under the hypothesis that D_d does *not* fulfill $(*)$. This yields characterizations of S_5 and the symplectic and unitary groups over the field with 2 elements.

Part III is the inductive part of the proof. In Part III it is assumed that D_d *does* fulfill $(*)$. This means that we can assume inductively that $[D_d]$ is known. In the most interesting case, we assume that $[D_d]$ is the $U_n(2)$-graph. If $n < 6$, then Γ turns out to be classical. If $n > 6$, this assumption leads to a contradiction. But $n = 6$ turns out to be just right, and we obtain the sporadic group Fi_{22}! Repeating the process, we obtain Fi_{23} and then Fi_{24}.

Let's suppose now that D is a set of 3-transpositions with $[D_d]$ isomorphic to the Fi_{24}-graphs for all $d \in D$. Let $e \in A_d$, let $R = D_d \cap D_e$ and let $x \in R$. Then $R_x = D_d \cap D_x \cap D_u$. Since $[D_x]$ is the Fi_{24}-graph, we know that $[R_x]$ is the disjoint union of three copies of the $\mathrm{O}_8^+(3)$-graph. Thus R is a set of 3-transpositions satisfying the hypotheses but not the conclusions of the Proposition above.

With this contradiction, we conclude that Fi_{25} does not exist!

We turn now to Moufang polygons. The classification of 3-transpositions is a problem in group theory, but the farther you go into the problem, the more graph theory takes over. The classification of Moufang polygons, by contrast, is a problem in graph theory to start with. There are many parallels, however, between the problem of classifying 3-transpositions and the problem of classifying Moufang *quadrangles*.

(1) They both involve the geometry of simple groups.

(2) In both cases, the simple groups involved are almost all classical (orthogonal, symplectic

and unitary) with just three exceptions.

(3) The classification of Moufang quadrangles can be divided into the same three parts: basic lemmas, an non-inductive part and then an inductive part. In Part I we show that the quadrangle Γ is determined by a sequence of four root groups $U_1, \ldots, U_4$ and the subgroup of $\mathrm{Aut}(\Gamma)$ that they generate. We call Γ *wide* if the commutator groups $[U_1, U_3]$ and $[U_2, U_4]$ are both non-trivial.

Fundamental Lemma: If Γ is wide, then there is canonically embedded subquadrangle Γ_0 which isn't wide.

In Part II, non-wide Moufang quadrangles are classified. In this case we have orthogonal examples coming from anisotropic quadratic forms and unitary examples coming from skew fields with involution (i.e. an anti-automorphism of order 2). There is also a third case, the *indifferent* quadrangles, where, in fact, $[U_1, U_3]$ and $[U_2, U_4]$ are both trivial; these exist only in characteristic 2.

In Part III, it is assumed that Γ is wide and that, inductively, the subquadrangle Γ_0 is known. The case that Γ_0 is an indifferent quadrangle leads to a contradiction. A skew-field with involution leads to more unitary groups. In the one remaining case, Γ_0 is the quadrangle associated to an anisotropic quadratic space (K, L, q).

If $\dim_K L \leq 4$, we obtain the last family of unitary groups. At this point the classification ought to be over (and it is if, for example, K is finite or a local field, since in these case there are no anisotropic quadratic forms of dimension greater than 4). Instead, three exceptional families arise, one after the other. These are the "exceptional" quadrangles of type E_6, E_7 and E_8.

The corresponding anisotropic quadratic spaces "of type E_6, E_7 and E_8" have dimension 6, 8 and 12. De Medts has characterized these quadratic forms in terms of classical invariants. For example, the quadratic forms of type E_7 are precisely the anisotropic forms of dimension 8 with trivial discriminant but non-trivial Clifford invariant. These exist, of course, only over certain fields K.

(There is, in fact, one more family of Moufang quadrangles, those of type F_4. These arise in the inductive situation when the bilinear form associated with the quadratic form q is allowed to be degenerate. These quadrangles exist only over certain fields of characteristic 2.)

In recent work with Bernhard Mühlherr and Holger Petersson, which is nearing completion, we are studying the exceptional Moufang quadrangles under the hypothesis that the field K is complete with respect to a discrete valuation. In this case there is a unique Bruhat-Tits building of type $\tilde{C}_2$ whose building at infinity is the exceptional Moufang quadrangle associated with a given quadratic form q of type E_6, E_7, E_8 or F_4 (by results of Tits and, in the E_8-case, Koen Struyve). Our result is a complete classification of all the possible residues of such a Bruhat-Tits building. This is the last step in a classification of the local structure of all Bruhat-Tits buildings of dimension at least 2. This work is closely connected with an old result of Springer.

If (K, L, q) is an anisotropic quadratic space over a field K complete with respect to a discrete valuation, then Springer observed that there are two "residual" anisotropic quadratic spaces $(\bar{K}, \bar{L}_0, \bar{q}_0)$ and $(\bar{K}, \bar{L}_1, \bar{q}_1)$ whose dimensions add up to $\dim_K L$. If, for example, the quadratic form q is the reduced norm of an octonion division algebra over K, equivalently, if q is an anisotropic quadratic form of dimension 8 with trivial discriminant and trivial Clifford invariant, then these two dimensions are either $8 + 0$ or $4 + 4$. This corresponds to the fact that if the building at infinity of a Bruhat-Tits building is a projective plane defined over an octonion division algebra, then the residues are projective planes defined either over an octonion division algebra with center $\bar{K}$ or over a quaternion division algebra with center $\bar{K}$. If the dimensions are $4 + 4$, then there exist, in general, separable quadratic splitting fields E/K of q which are ramified and others that are not. (We are ignoring the additional possibility in this discussion that if the characteristic of $\bar{K}$ is 2, the bilinear form associated with $\bar{q}_0$ or $\bar{q}_1$ can be identically zero.)

If, in contrast, q is of type E_7, in other words, if q is an anisotropic quadratic form of dimension 8 with trivial discriminant and non-trivial Clifford invariant, then the residual dimensions can be $8 + 0$, $6 + 2$ or $4 + 4$. In the first two cases, all the separable quadratic splitting fields E/K of q are unramified; in the third, they are either all unramified or all ramified. Furthermore, the 6- and 8-dimensional residue forms are, generically, quadratic forms of type E_6 or E_7 over $\bar{K}$, but when $\bar{K}$ has characteristic 2, they can also be quadratic forms of type F_4. All of these cases correspond to different configurations of local structure in the corresponding Bruhat-Tits buildings.

Without going into any more details, we hope that our results make it clear that the quadratic forms of type E_6, E_7, E_8 and F_4 deserve a prominent place—alongside the norms of quaternion and octonion division algebras—in the theory of low dimensional quadratic forms and their connections to exceptional groups.

A theme of this talk has been that the shrewd use of graph theory leads to many very nice results in the theory of groups. We all owe an enormous debt to Bernd Fischer for showing us just how beautiful these kinds of results can be.