Simple groups of Lie type without Lie theory

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For some years now I have been trying to understand the finite simple groups of Lie type, in a way that (a) explains their covering groups, and (b) is in the smallest possible dimension, to facilitate concrete calculations. I should say I'm really only interested in definitions, not theorems.

The usual approach via Lie algebras and/or algebraic groups, while having the advantage of uniformity, does not do this.

1 Lie groups

First we should understand the much simpler situation of the complex Lie groups. There are three infinite families of classical groups (orthogonal = real, unitary/linear = complex, and symplectic = quaternionic), and five exceptional groups (G_2, F_4, E_6, E_7, E_8). The Lie algebras of the classical groups are best described as algebras of matrices, and have dimension of the order of n^2 or $\frac{1}{2}n^2$, where the natural representation has degree n.

The exceptional Lie algebras have dimension respectively 14, 52, 78, 133, 248, while the smallest representations have dimension respectively 7, 26, 27, 56, 248.

The 7-dimensional representation of G_2 is on the pure imaginary octonions (Cayley numbers): indeed G_2 is the automorphism group of this Cayley algebra.

The 26-dimensional representation of F_4 is similarly the automorphism group of the exceptional Jordan algebra (or Albert algebra). This algebra consists of 3×3 Hermitian matrices over the Cayley numbers, with multiplication defined by $X \circ Y = \frac{1}{2}(XY + YX)$. The group can be extended to E_6 by preserving only the (Freudenthal) determinant, and not the algebra product.

The compact real forms (e.g. for orthogonal groups this means the underlying quadratic form is positive definite) reveal that

- F_4 acts on a 26-dimensional real (orthogonal) space;
- E_6 acts on a 27-dimensional complex (unitary) space;
- E_7 acts on a 28-dimensional quaternionic (symplectic) space.

This suggests the best way to study these groups is top-down, from the 28-dimensional representation of E_7, rather than the traditional bottom-up approach starting from the exceptional Jordan algebra for F_4.

2 Finite groups of Lie type

In the finite field case, the situation is more complicated. There are six families of classical groups (linear, unitary, symplectic, and three families of orthogonal groups), and ten families of exceptional groups:

Chevalley (1955)		$G_2(q)$	$F_4(q)$	$E_6(q)$	$E_7(q)$	$E_8(q)$
Steinberg–Tits (1959)		${}^3D_4(q)$		${}^2E_6(q)$		
Suzuki–Ree (1961)	${}^2B_2(2^{2n+1})$	${}^2G_2(3^{2n+1})$	${}^2F_4(2^{2n+1})$			

As well as the problems of the large dimension of the Lie algebra, there is also the problem of disappearing scalars: there is a double cover $2E_7(q)$ for all odd q, a triple cover $3E_6(q)$ whenever $q \equiv 1 \bmod 3$, and $3.{}^2E_6(q)$ whenever $q \equiv 2 \bmod 3$. Let's look at the situation in detail:

$\mathbb{H}$ symplectic 4-form	$\mathbb{C}$ linear, 3-form	unitary	$\mathbb{R}$ real algebras	$\mathbb{W}$ 'double algebras'	
			$E_8(q)$ Lie algebra dim. 248		$\mathbb{O}$ octonions dimension 8
$(2)E_7(q)$ dim. 28	$(3)E_6(q)$, Dickson form $\approx$ determinant dim. $27 = 3^3$	$(3)^2E_6(q)$	$F_4(q)$ Jordan algebra dim. 26	${}^2F_4(2^{2n+1})$	$\mathbb{H}$ quaternions dimension 4
		${}^3D_4(q)$ Springer algebra twisted octonion dim. $8 = 2^3$	$G_2(q)$, Cayley algebra dim. 7	${}^2G_2(3^{2n+1})$	$\mathbb{C}$ complexes dimension 2
				${}^2B_2(2^{2n+1})$ dim. 4	

3 History

Of these, G_2, F_4 and E_6 are well-studied, and well-understood in terms of the Cayley (octonion) algebra and the Albert (Jordan) algebra. We'll start with these and work outwards to the rest. Historically, Dickson (1901) constructed G_2 in odd characteristic (characteristic 2 in 1905) essentially as automorphisms of the Cayley numbers; and E_6 as the stabilizer of a cubic form of 45 terms in 27 variables. If the variables are called $x_i, y_i, z_{ij} = -z_{ji}$ for $i \neq j \in \{1,2,3,4,5,6\}$ then the form is $\sum_{i<j} x_i y_j z_{ij} + \sum z_{ij} z_{kl} z_{mn}$ where the second sum is over all even permutations.

Jacobson (1959–1961) studied F_4 and E_6 in characteristic not 2 or 3, using Jordan algebras, and Springer introduced twisted octonion algebras to study ${}^3D_4(q)$. Suzuki's construction of his group around the same time was already as 4×4 matrices. R. B. Brown (1969) studied E_7 in characteristic not 2 using a quartic form with 1036 terms in 56 variables.

4 Root systems

The root systems of type B_2, G_2, F_4 and E_8 are conveniently thought of as subsets of (respectively) $\mathbb{C}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. The short roots of B_2, G_2, F_4 are respectively

$$\begin{array}{l} \pm 1, \pm i \\ \pm 1, \pm\omega, \pm\overline{\omega} \\ \pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k) \end{array} \tag{1}$$

and the long roots are

$$\begin{array}{l} \pm 1 \pm i \\ \pm\omega^a(1-\omega) \\ \pm 1 \pm i, \ldots \end{array} \tag{2}$$

5 Algebras

Look first at the column of algebras. The Lie algebra is spanned by 240 root vectors e_r, and 8 dimensions of nilpotent elements h_r, and the Lie product is defined by equations like

$$\begin{array}{rcl} [e_r, e_s] &=& \pm e_{r+s} \\ [e_r, h_s] &=& \lambda_{r,s} e_r \\ [h_r, h_s] &=& 0. \end{array} \tag{3}$$

The (split) Cayley algebra may be spanned by 6 short root vectors e_r, corresponding to the short roots of the G_2 root system, and two orthogonal idempotents $e_{\pm 0}$, with product defined by

$$\begin{array}{rcll} e_r e_s &=& \pm e_{r+s} & (r, s \neq \pm 0) \\ e_z e_r &=& \lambda_{z,r} e_r & z = \pm 0 \end{array} \tag{4}$$

The (split) Jordan algebra may similarly be spanned by 24 short root vectors e_r, corresponding to the short roots of F_4, and three orthogonal idempotents e_0, $e_{\omega 0}$, $e_{\overline{\omega}}$. The Jordan product is given by equations of the form

$$\begin{aligned} e_r \circ e_s &= e_{r+s} \\ e_z \circ e_r &= \lambda_{z,r} e_r \end{aligned} \tag{5}$$

In every case there is also a bilinear form such that $e_r.e_{-r} = \pm 1$.

6 Double algebras

The map ϕ defined on B_2, G_2 and F_4 respectively by

$$\begin{aligned} z &\mapsto (1+i)\overline{z} \\ z &\mapsto (1-\overline{\omega})\overline{z} \\ z &\mapsto (1+i)z^j \end{aligned} \tag{6}$$

maps short roots to long roots and squares to $p = 2, 3, 2$ respectively.

Now take the above (Jordan and Cayley) algebras over a field of characteristic p and odd degree, and reduce the dimension by 1 by defining $e_0 + e_{-0} = 0$ in the Cayley algebra, and $e_0 + e_{\omega 0} + e_{\overline{\omega} 0} = 0$ in the Jordan algebra. Similarly for B_2, take the 4-space spanned by $e_{\pm 1}$ and $e_{\pm i}$, with just the bilinear form, and zero product.

We can define a new product by

$$e_r \star e_s = \pm e_{\phi^{-1}(r+s)}$$

(and other things involving the zeroes), and extending biadditively and

$$(\lambda u) \star v = u \star (\lambda v) = \lambda^\sigma (u \star v)$$

where $\lambda^{\sigma^2} = \lambda^{-p}$, and then restricting $\star$ to expressions

$$\sum (\lambda_{r,s} e_r) \star e_s \text{ such that } \sum \lambda_{r,s}(e_r \times e_s) = 0.$$

Then it turns out that the Suzuki and Ree groups are exactly the automorphism groups of these double algebras ($\mathbb{W}$-algebras). As they are in the $\mathbb{R}$ column of the main table, I might call them $\mathbb{RW}$-algebras.

7 Quaternionic E_7

The roots of E_7 are perhaps best described as the $14 + 112 = 126$ pure imaginary octonions $\pm i_0, \ldots, \pm i_6$ (that is i_t for $t \in \mathbb{F}_7$), and $\frac{1}{2}(\pm i_2 \pm i_4 \pm i_5 \pm i_6)$ and images under $t \mapsto t + 1$. The minimal vectors of the dual lattice are $\pm i_t \pm i_{t+1} \pm i_{t+3}$.

Label these pairs $\pm v$ by $H_{-t}, I_{-t}, J_{-t}, K_{-t}$, where the signs are respectively $+++$, $+--$, $-+-$, $--+$. These are quaternions, and we now construct 63 root groups $SU(2)$ as follows: the root group corresponding to $\pm i_0$ is obtained by letting a quaternion $q = z + wj$ act as $\begin{pmatrix} z & wj \\ wj & z \end{pmatrix}$ on (H_1, I_1), (H_2, J_2), (H_4, K_4) and as $\begin{pmatrix} \bar{z} & \bar{w}j \\ \bar{w}j & z \end{pmatrix}$ on (J_1, K_1), (K_2, I_2) and (I_4, J_4). The other orbit of root groups is similar: the first matrix acts on (H_2, I_6), (H_4, J_5) and (H_1, K_3), and the second on (H_5, I_4), (H_3, J_1), (H_6, K_2).

These matrices generate (the double cover of the compact real form of) E_7. To get this as a 56-dimensional complex representation, just write $q = q' + q''j$ for each quaternion q. Then the quartic form is a sum over orbits of the Weyl group:

$$-\frac{1}{4}\sum (H_0'H_0'')^2 + \frac{1}{2}\sum H_0'H_0''I_0'I_0'' + \sum H_0'I_0'J_0'K_0'$$

This can be interpreted over any field of characteristic not 2, and gives a definition of $2E_7(q)$ for any odd q. To get $E_7(q)$ in characteristic 2, just delete the terms which don't make sense, and just keep $\sum H_0'I_0'J_0'K_0'$. (See my webpage for a preprint with more details.)