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Abstracts of Plenary and Invited Lectures

Section:

0. Plenary Lectures

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Geometric model theory

A confluence of two streams of model theory has led in recent years to some interesting connections with other subjects, including finite group theory and algebraic geometry. I will attempt to survey these two streams and explain the nature of the contact. The main ingredients are:
1) Algebraic model theory: model completeness theorems .
2) A general theory of finite (and infinite) dimensional geometries, of a certain type. (Stability and its generalizations.)
3) Characterization theorems for classical geometries within the abstract class considered in (2).
The prototype for model completeness theorems is Tarski's theorem relating to real algebraic geometry. It asserts that the class of "semi-algebraic sets" - subsets of $\mathbb{R}^n$ defined by polynomial equalities and inequalities - is closed under projections from $\mathbb{R}^n$ to $\mathbb{R}^m$. it follows that all basic elementary operations (including, for instance, taking the closure of a set) can be carried out without leaving this class. As a corollary, an algorithm exists to determine in advance the outcome of any elementary geometric construction; and one is guaranteed that such a construction will never lead to topological or set theoretic pathologies.
Similar results have been proved (i. a.) for p-adic algebraic geometry, for geometries incorporating analytic functions (on bounded domains, and some others), for the ring of algebraic integers. The present applications use the model completeness results for fields with differential and difference operators. (These results are due to Robinson, Ax, Kochen, Macintyre, Van den Dries, Denef, Wilkie, and others.)
The theory of finite dimensional geometries referred to in (2) begins with the conclusion of the model completeness results, without assuming however any particular algebraic structure. One is given abstractly a class of "definable" sets (corresponding to the semi-algebraic sets in Tarski's theorem) and a dimension theory on the definable sets.
The theory generalizes basic features of algebraic geometry, including algebraic groups and homogeneous spaces; in this it generalizes the differential Galois theory of Lie and Kolchin. It allows also some phenomena that do not occur classically, in particular for the mixing of geometries of essentially different types. Shelah's theory shows how to analyze a given structure in

terms of certain simple geometries within it, and describes their possible interactions.
(3) Ordinary algebraic geometry over algebraically closed fields is entirely characterized, within this class of abstract geometries, by a nondegeneracy and nonlinearity condition and a condition on the dimensions of intersections (generally, intersecting a k-dimensional set with one of codimension 1 should give a set of pure dimension $k-1$.) This result of Zil'ber and the author appeared initially as a foundational result; curiously it turned out to be a main component in the applications.
The connection with geometry is made by considering algebraic varieties with additional structure; here we will take it to be a vector field. Similar theorems hold for several commuting fields, or for correspondences (the latter required an extension of stability theory.)
We will illustrate the theory using a theorem placing a limit on the number of integral subvarieties of the flow. It is generally felt that algebraic integral varieties of sufficiently general flows should be rare, but their analysis is difficult and general results are few. (Except in codimension one, where the qualitative situation is understood thanks to results of Jouanolou). The approach here is to consider these varieties as the closed sets of an abstract geometry, and to use model theoretic analysis of such geometries, The model theory then identifies the exceptional cases, where many integral subvarieties do exist, and describes dividing lines that must be taken into account in any general description of algebraic ODE's and their transformations. Related results have been used to draw consequences in diophantine geometry.
Let V be a smooth complex algebraic variety, with an algebraic vector field ξ. We will be interested in algebraic families $\{U\}$ of algebraic subvarieties V, that are left invariant by the flow corresponding to ξ. We do not demand that U itself be left invariant; only that the deformations of U remain in the same family. Call a subvariety U belonging to such a family, ξcoherent. Thus every point is ξ-coherent, as well as every integral subvariety of ξ. One defines a "geometry" on V, roughly speaking by taking the "closed sets" of X (or of X^n) to be the coherent subvarieties of X or X^n.
By a "curve", let us mean an infinite ξ-coherent subvariety of V, with no proper subvarieties of the same kind. It may well have higher dimension than 1, as an algebraic variety. Through each point there passes at least one "curve"; there may be infinitely or uncountably many. If there are more than $n = \dim(V)$ "curves" through every (sufficiently general) point of V, we will say that the Kolchin geometry is nondegenerate.
If $\xi = 0$, every subvariety is integral, and we simply have the usual algebraic geometry on V. More generally if ξ admits an algebraic first integral, at least a part of the geometry is classical. The conclusion of (3) will help identify such cases.
Theorem Assume the Kolchin geometry on V is nondegenerate. After removing from V a finite number of lower dimensional integral subvarieties, one of the following occurs:

a. ξ has an algebraic first integral; i. e. there exists a map $f : V \to W$, W an algebraic variety of dimension ≥ 1, such that the vector field ξ is parallel to the fibers of f.
b. There exists an equivariant map $f : V \to V'$, V' an algebraic variety of smaller dimension carrying a vector field ξ', such that the fibers of f are principal homogeneous spaces for algebraic groups; and the action respects the vector field.
c. There exists a map f as in (3), such that the fibers are rational images of Abelian varieties under a finite map; the vector field corresponds to the Manin equation.
The equations occurring in (3) have an interesting structure that is precisely described by the model theory; it is this description that has been used in applications to diophantine geometry.

The proof has the following stages. Quantifier elimination for differential fields (and some additional model theory for these fields) shows that the abstract geometry corresponding to the coherent subvarieties enjoys the basic properties necessary to apply (2). The consequent dimension theory will be bounded by, but will not in general coincide with, the original dimension on the variety. Shelah's theory will be used to find a canonical filtration by maps $f : V \to W$, such that the fibers at each stage are coordinatized by one-dimensional geometries. The situation then splits according to the the structure theorems for these simple geometries, and one obtains an abstract version of the case division. At this point one recalls the differential geometric provenance of the geometry, and deduces (a-c).
Shelah's theory is actually more general than what is indicated here, and does not assume *finite* dimensionality. This greater generality should be of use in future applications, in particular to PDEs.
On the other hand, restrictive assumptions are currently made on the nature of the dimension theory. These are valid in complex algebraic geometry, but rule out e. g. p-adic geometry. Indeed the theory did not initially apply to difference equations; stability needed to be generalized to a wider context, simple theories (Shelah, Kim, Pillay, Hart, Wagner, Buechler, Shami, and others). A separate theory (O-minimality) exists for geometries resembling the reals, and in particular the analog of (3) has been proved by Peterzil and Starchenko. But a satisfactory common framework remains a challenge.
Future applications will depend on expansions of the current frontier of model completeness results, perhaps covering for example fields with global aspects such as theories of heights, and on extensions of the pure model theory of geometric structures.
(References in the article text.)