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Abstracts of Plenary and Invited Lectures

Section:

0. Plenary Lectures

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Macdonald, I. G., Queen Mary and Westfield College, University of London, England

Constant Term Identities, Orthogonal Polynomials and Affine Hecke Algebras

Constant term indentities

The archetype of the constant term identities to be considered is the following statement, first conjectured in 1961 by F. J. Dyson but proved soon afterwards. Let $x_1, \ldots, x_n$ be independent indeterminates and k a positive integer. Then the constant term in the Laurent polynomial

$$\Delta_k(x) = \prod_{\substack{i \neq j \\ i,j=1}}^{n} \left(1 \frac{x_i}{x_j}\right)^k \tag{1}$$

is equal to $(nk)!/(k!)^n$.

Later, in 1975, G. E. Andrews proposed an extrapolation of (1) involving an extra parameter q. We shall use the notion

$$(x;q)_k = \prod_{i=0}^{k-1} \left(1 - q^i x\right).$$

Clearly, as $q \to 1$ we have $(x;q)_k \to (1x)^k$, and $(q;q)_k/(1-q)^k \to k!$

Let

$$\Delta_k(x;q) = \prod_{\substack{i \neq j \\ i,j=1}}^{n} \left(\varepsilon_{ij} \frac{x_i}{x_j}; q\right)_k \tag{2}$$

where $\varepsilon_{ij} = 1$ or q according as $i < j$ or $i > j$. Then Andrews conjectured that the constant term of (2) (i. e. , involving q but none of the x_i) should be

$$(q;q)_{nk}/(q;q)_k^n. \tag{3}$$

Clearly (2) reduces to (1) as $q \to 1$, so that Andrews' conjecture includes Dyson's.

From our point of view, the polynomial $\Delta_k(x;q)$ is to be regarded as attached to the simple Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ consisting of the $n \times n$ complex matrices with trace zero. In fact each finite-dimensional complex simple Lie algebra $\mathfrak{g}$ gives rise to an analogous Laurent polynomial and constant term identity.

Orthogonal polynomials

Let Λ_n denote the ring of symmetric polynomials in $x_1, \ldots, x_n$ with coefficients in the field $F = \mathbb{Q}(q)$. For each partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ of length $\leq n$, let m_λ denote the monomial symmetric function indexed by λ, that is to say the sum of all distinct monomials obtainable from $x_1^{\lambda_1}, \ldots, x_n^{\lambda_n}$ by permuting the x_i, Clearly the m_λ form an F-basis of Λ_n.

We shall use the polynomial $\Delta_k = \Delta_k(x;q)$ to define a scalar product on Λ_n, as follows: if $f, g \in \Lambda_n$ then

$$\langle f, g \rangle_k = \text{ constant term in } f\bar{g}\Delta_k$$

where $\bar{g}$ is obtained from g by replacing each x_i by x_i^{-1}. One shows then that there is a unique basis (P_λ) of Λ_n, indexed by partitions of length $\leq n$, such that

$$\text{(a)} \qquad P_\lambda = m_\lambda + \sum_{\mu < \lambda} a_{\lambda\mu} m_\mu$$

with coefficients $a_{\lambda\mu} \in F$, where $\mu < \lambda$ means that μ precedes λ in the lexicographic ordering; and

$$\text{(b)} \qquad \langle P_\lambda, P_\mu \rangle = 0 \quad \text{whenever } \lambda \neq \mu.$$

Moreover, there is a closed formula for the squared norm of P_λ, namely

$$\langle P_\lambda, P_\lambda \rangle_k = \frac{(t;t)_n}{(1t)^n} \prod_{\substack{i<j \\ i,j=1}}^{n} \frac{(q^{\lambda_i - \lambda_j} t^{j-i}; q)_k}{(q^{\lambda_i \lambda_j} t^{j-i}; q^{-1})_k} \tag{4}$$

where $t = q^k$. In particular, when $\lambda = 0$ we have $P_\lambda = 1$, and (4) then reduces to the constant term (3) of Δ_k.

Again, all this can be done in the context of an arbitrary finite-dimensional simple Lie algebra $\mathfrak{g}$ in place of $\mathfrak{sl}_n$, and thus for each such $\mathfrak{g}$ we have a family of orthogonal polynomials P_λ, symmetric under the Weyl group of $\mathfrak{g}$ and indexed by the dominant weights.

Affine Hecke algebras

In the case of $\mathfrak{sl}_n$ the norm formula (4) can be proved directly. For arbitrary $\mathfrak{g}$ there is an analogous formula which was first proved in full generality by Cherednik. Cherednik's proof uses the affine Hecke algebra $\mathcal{H}$ attached to $\mathfrak{g}$ and its action on the space spanned by the polynomials P_λ; this leads to a family of commuting operators on this space whose simultaneous eigenfunctions are precisely the P_λ. The norm formula (4) – or rather its counterpart for arbitrary $\mathfrak{g}$ – can then be established by induction on k.