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1. Logic

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Generic Large Cardinals: New Axioms for Mathematics?

While the standard axiomatization of mathematics, (*Zermelo-Fraenkel Set Theory with the Axiom of Choice*, ZFC) has been extremely successful in resolving the foundational issues that arose at the turn of the century, it has some glaring shortcomings. These shortcomings are largely due to their inability to settle certain natural problems.
Most of these problems are of an infinitary character (though many can be coded as questions about the solutions to various diophantine equations). Most prominent among these problems are Hilbert's 1^{st} problem (the Continuum Hypothesis), and issues having to do with the use of the Axiom of Choice. The development of Forcing, in the early 1960's, led to independence results in most areas of mathematics that have a strong infinitary content, particularly including measure theory and other parts of analysis, infinite group theory, topology and combinatorics.
In a program encouraged by Gödel, attempts were made to find natural extensions of ZFC that were strong enough to settle these questions. (Of course, Gödel's incompleteness theorems prevent the development of a *complete* axiomatization.) In these efforts, two competing axiom systems arose: *Determinacy Axioms* and *Large Cardinal Axioms*. Each system has its merits: Determinacy Axioms answer most questions about definable, or constructible sets, and Large Cardinal Axioms are natural closure and symmetry properties of the mathematical universe. However, on the surface the two systems have very little to do with each other.
Much work was done exploring the connections between these two axiom systems, culminating in a series of theorems in the late 1980's showing that they are inextricably intertwined. An apparently technical device that arose in the course of this work was the notion of a *Generic Large Cardinal* or generic elementary embedding. These assumptions eventually turned out to be natural generalizations of various Large Cardinal Axioms and have the virtue that they settle the Continuum Hypothesis, imply the Determinacy axioms and can be viewed as strong symmetry principles.
Because of the close link between Forcing and Boolean Algebras, these axioms are most conveniently expressed as properties of ideals on Boolean Algebras. Ulam, in 1930, proposed the axiom of *Measurable Cardinals*, i. e. the existence of a set X and a countably complete, non-atomic, two valued measure μ defined on all subsets of X. This can be rephrased as

the existence of a countably complete prime ideal on the Boolean Algebra of subsets of X that contains all of the singletons. Ulam showed that such a set X must have cardinality much larger than the real numbers or the collection of subsets of the real numbers.
Gitik and Shelah extended this by showing that there cannot be a countably complete ideal I on the collection of subsets of the real numbers such that the quotient Boolean Algebra has a dense countable subset (and similar results for larger sets than the real numbers). Thus the strongest ideal property one can hope is consistent is an ideal such that the quotient Boolean algebra has a dense set of cardinality $\aleph_1$ (the first uncountable cardinal). Such ideals are called $\aleph_1$-*dense*.
Some examples of theorems of the author discussed in the talk include:

Theorem *If there is a normal, fine $\aleph_1$-dense ideal on the subsets of $P(\mathbb{R})$ of cardinality $\mathbb{R}$ then*:

-The Continuum Hypothesis holds
-There is a Suslin Line
-Kurepa's Hypothesis Fails
-$\Box_{\omega_1}$ fails.
-$\omega_2 \rightarrow (\omega_1^2, \alpha)$ for all $\alpha < \omega_2$ (joint with Hajnal)
-A. D. holds in $L(\mathbb{R})$ (Heavily using results of Woodin)

(The hypothesis of this theorem are chosen for clarity; it is known they are far from optimal. Woodin has improved the CH result.) To transfer some of these properties to bigger cardinals it is possible to show, for example:

Theorem Suppose for all regular λ, and all natural numbers n, there is a normal fine ideal on $[\lambda^{+n}]^{\aleph_{n+1}}$ and the Continuum Hypothesis holds, then the Generalized Continuum Hypothesis holds.

The exact relationship of these ideal properties to large cardinals still retains many challenging open problems. However, it is possible to show that they form a hierarchy of consistency strength that sometimes directly meshes with the consistency strength of large cardinals. Typical theorems include:

Theorem Suppose that there is a normal, fine ideal I on $[\omega_3]^{\omega_2}$ and a set $X \in \breve{I}$ and $Y, Z \in V$ such that whenever $j : V \rightarrow M$ is the generic embedding of V into its ultrapower, we know $j``\omega_3 = Y$ and $j(X) = Z$, then there is an inner model with a huge cardinal

The hypothesis of the this theorem can be sharpened to a form of Chang's Conjecture with an additional combinatorial property. In the other direction the best known result for $\aleph_1$-dense ideals is:

Theorem Suppose that there is a pair of almost huge cardinals κ_0, κ_1 and an almost huge embedding j with $j(\kappa_0) = \kappa_1$. Then it is consistent that there is a uniform, countably complete $\aleph_1$-dense ideal on $\aleph_2$.

Finally, the talk will discuss the role played by generic large cardinal embeddings in showing reflection properties and their relationship to various squares. An important example of this is the following joint theorem of the author, Cummings and Magidor:

Theorem Suppose it is consistent with ZFC that there are infinitely many supercompact cardinals. Then it is consistent with ZFC that every stationary subset of $\aleph_{\omega+1}$ reflects and the property $\square_{\aleph_{\omega+1},\omega}$ holds.