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Abstracts of Plenary and Invited Lectures

Section:

1. Logic

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When is an equivalence relation classifiable?

In the works of Mackey [7], Glimm [2], and Effros [1], one finds the following notion of classifiability: An equivalence relation E is *classifiable* if the quotient object, X/E, consisting of all the equivalence classes, has a *nice* Borel structure.

More precisely, given an equivalence relation E on a standard Borel space X, let us follow terminology from analysis and say that E is *smooth* if the set of all equivalence classes $\{[x]_E : x \in X\}$ equipped with the σ-algebra generated by the projections of the E-invariant Borel sets may be realized as a subspace of a standard Borel space – or, equivalently, E is reducible to the equality relation on the reals in the sense that there is a Borel $f : X \to \mathbb{R}$ such that $x_1 E x_2$ if and only $f(x_1) = f(x_2)$.

This is the paradigm of classifiability that one finds implicit in much of the theory of infinite dimensional group representations and Ornstein's celebrated work in [8] on entropy in the context of Bernoulli shifts. It has been subjected to intense investigation following papers of Glimm, Effros, and then much later Harrington, Kechris, and Louveau, and is to a large degree well understood.

Yet despite the seductive charm of this wonderfully simple idea, it also true that this notion of classification does *not* match a great variety of examples from mathematical practise. For instance, [6] presents the Ulm invariants for countable abelian p-groups as a scheme of classification, and yet the equivalence relation of isomorphism on countable abelian p-groups is not smooth in the above sense. Similarly the analysis from [3] of discrete spectrum measure preserving transformations provides complete invariants for an equivalence relation that is not reducible to the equality relation on the reals. Finally, close to the very concerns that gave rise to the original definition, one finds in the spectral theorem – and its classification of unitary operators by measures considered up to absolute continuity – a classification theorem for a quotient object that *does not* admit such a "nice Borel structure".

In my talk I will describe some recent work in descriptive set theory which relates to a notion of classification far more generous than the smooth/nonsmooth dichotomy. In the last few years it has been possible to develop an understanding of when an equivalence relation admits classification by countable discrete structures (considered up to isomorphism) – or, equivalently, is reducible to the equality relation on the hereditarily countable sets.

References

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