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2. Algebra

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Geometry of Infinitesimal Group Schemes

The cohomology and representation theory of infinitesimal group schemes over a field of positive characteristic have fascinating relationships and geometric structures. Earlier studies, undertaken jointly with Brian Parshall, primarily concerned infinitesimal group schemes of height 1 (which have a more familiar interpretation as finite dimensional restricted Lie algebras). Many of the results of E. Friedlander and B. Parshall for restricted Lie algebras are modeled on analogous results for finite groups, especially algebro-geometric structures introduced by Daniel Quillen and Jon Carlson. More recent joint work with Andrei Suslin and Christopher Bendel has achieved a dramatic extension of these results to infinitesimal group schemes of arbitrary height and provided sharper results in the height 1 case.
Let k be a field of characteristic $p > 0$. A finite group scheme G over k is a finite dimensional cocommutative Hopf algebra A_G with (commutative) dual Hopf algebra $k[G]$. Such a finite group scheme G is said to be infinitesimal if $k[G]$ is a local algebra. The height of an infinitesimal group scheme G is the minimum integer r such that the maximal ideal of $k[G]$ is generated by elements whose p^r-power equals 0. If G is an infinitesimal group scheme of height 1, then A_G equals the restricted enveloping algebra of $\mathcal{G} = \mathcal{L}\rangle\rceil(\mathcal{G})$.
Infinitesimal group schemes arise in the study of the (rational) representation theory of affine algebraic groups K over k. The r-th Frobenius kernel of K, $K_{(r)}$, is defined as the scheme-theoretic kernel of the the r-th iterate of Frobenius $\phi^r : K \to K^{(r)}$ (where $K^{(r)}$ is the Frobenius-twisted form of K, isomorphic to K whenever K is defined over the finite field of p^r elements). So defined, $K_{(r)}$ is an infinitesimal group scheme of height r. Unlike the situation in characteristic 0, the representation theory of the algebraic group K is not faithfully reflected in the representation theory of $Lie(K)$; on the other hand, the study of all the "infinitesimal neighborhoods of the identity" of K (i. e., the study of the representation theory of all $K_{(r)}$) often does faithfully reflect the representation theory of K.
(Rational) representations of an affine group scheme G are defined to be comodules for the Hopf algebra $k[G]$; if G is a finite group scheme, then such representations are equivalent to modules for A_G. One central aspect of the representation theory of G is the study of the Ext-groups $Ext^i_G(M, N)$ where M, N are rational G-modules. In particular, the cohomology algebra

$H^*(G,k) = Ext^*_G(k,k)$ contains significant information about G and its representations.

The geometric nature of the representation theory of height 1 infinitesimal group schemes is reflected in the following theorem of E. Friedlander and B. Parshall, with bound on p improved by Henning Andersen and Jens Jantzen. If K is a semi-simple algebraic group over k with Coxeter number of each factor less than p, then the cohomology algebra $H^*(K_{(1)},k)$ is concentrated in even dimensions and equals the coordinate algebra of the variety of nilpotent matrices in $Lie(K)$ [1], [2].

A key theorem of E. Friedlander and A. Suslin is the following: if G is a finite group scheme over k, then $H^*(G,k)$ is finitely generated [5]. If G is either discrete (i. e., equivalent to the group algebra of a finite group) or infinitesimal of height 1, then finite generation is well known; the proof of this result even for height 2 infinitesimal groups schemes has required new techniques and considerable effort. For G an infinitesimal group scheme of height r, E. Friedlander and A. Suslin in collaboration with C. Bendel identified $H^*(G,k)$ modulo p^r-powers. Stated geometrically, the spectrum of $H^*(G,k)$ is the variety of all infinitesimal 1-parameter subgroups of G (i. e., morphisms $G_{a(r)} \to G$) [6], [7]. In the special case $r = 1$, this is a sharper form of a theorem of J. Jantzen.

Following work of Jon Carlson, George Avrunin, and Leonard Scott in the context of representation of finite groups, E. Friedlander and B. Parshall (using J. Jantzen's theorem mentioned above) proved the following result. Let G be an infinitesimal group scheme of height 1 and let M be a finite dimensional (rational) G-module. Then the variety of the annihilator ideal inside the symmetric algebra on $\mathcal{G}^*$, $S^*(\mathcal{G}^*)$, for a natural action of $S^*(\mathcal{G}^*)$ via $H^{ev}(G,k)$ on $Ext^*(M,M)$ can be identified with the subvariety of $\mathcal{G} = \mathcal{L}\rangle\rceil(\mathcal{G})$ consisting of elements $X \in \mathcal{G}$ whose p-th power is 0 and for which M restricted to $\langle X\rangle$ is injective, where $\langle X\rangle$ is the cyclic Lie algebra generated by X [3], [4]. Once again, A. Suslin, E. Friedlander, and C. Bendel sharpened and generalized this result to an infinitesimal group scheme G of arbitrary height r; a description of the variety of the annihilator ideal of the commutative algebra $H^{ev}(G,k)$ acting on $Ext^*(M,M)$ is given in terms of the restrictions of the rational G-module M to infinitesimal 1-parameter subgroups of G [7].

Infinitesimal 1-parameter subgroups of G play a role very analogous to (shifted) elementary abelian subgroups of a finite group. One challenging problem for future study is to develop methods which apply to all finite group schemes and which specialize to give known results for finite groups and infinitesimal group schemes. Another less well defined challenge is to usefully employ more subtle invariants of rational representations which reflect finer local information than merely the injectivity of the representation.

References

[1] E. Friedlander and B. Parshall, *On the cohomology of algebraic*

groups and related finite groups, Inventiones Math., **74** (1983), 85-117

[2] E. Friedlander and B. Parshall *Cohomology of Lie algebras and algebraic groups* , Amer. J. Math. **108** (1986), 235-253

[3] E. Friedlander and B. Parshall *Geometry of p-Unipotent Lie algebras* , J. Algebra **109** (1987), 25-45 (1987)

[4] E. Friedlander and B. Parshall *Support varieties for restricted Lie algebras* , Inventiones Math. **86** (1986), 553-562 (1986)

[5] E. Friedlander and A. Suslin *Cohomology of finite group schemes over a field* , Inventiones Math. **127** (1997) 209–270

[6] A. Suslin, E. Friedlander, and C. Bendel *Infinitesimal 1-parameter subgroups and cohomology* , Journal of the American Mathematical Society **10** (1997), 693 – 728

[7] A. Suslin, E. Friedlander, and C. Bendel *Support varieties for infinitesimal group schemes* , Journal of the American Mathematical Society **10** (1997), 729 – 759