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Section:

2. Algebra

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On the Burnside problem for groups of even exponent

Recall that the notorious Burnside problem (posed in 1902, see [B]) asks whether any finitely generated group that satisfies the law $x^n \equiv 1$ (n is a fixed positive integer called the *exponent* of G) is necessarily finite. This is proven only for exponents $n \leq 4$ and $n = 6$ (the case $n = 3$ is due to Burnside, $n = 4$ to Sanov, and $n = 6$ to Hall, see [MKS]). A negative solution to the Burnside problem (for odd exponents) was given by Novikov and Adian [NA] in 1968 who proved the existence of infinite m-generator groups with $m \geq 2$ of any odd exponent $n \geq 4381$ (a simpler geometric solution to this problem for odd $n > 10^{10}$ was later given by Ol'shanskii [Ol]; note that Adian [Ad] later improved on the original estimate to odd $n \geq 665$). We remark that attempts to approach the Burnside problem via finite groups gave rise to a restricted version of the Burnside problem [M] that asks whether there exists a number $f(m,n)$ so that the order of any finite m-generator group of exponent n is less than $f(m,n)$. The existence of such a bound $f(m,n)$ was proven for prime n by Kostrikin [K] and for $n = p^\ell$ with prime p by Zelmanov [Z1]-[Z2].
However, the Burnside problem for even exponents n without odd divisor ≥ 665, especially interesting for $n = 2^\ell \gg 1$, remained open. The principal difference between odd and even exponents in the Burnside problem can be illustrated by pointing out that, on the one hand, for every odd $n \gg 1$ there are infinite 2-generator groups of exponent n all of whose proper subgroups are cyclic and, on the other hand, any 2-group the orders of whose abelian (or finite) subgroups are bounded is itself finite.
A negative solution to the Burnside problem for even exponents $n \gg 1$ given in speaker's article [Iv] is based on the following inductive construction (which is analogous to Ol'shanskii's construction [Ol] for odd $n > 10^{10}$). Let F_m be a free group over an alphabet $\mathcal{A} = \{a_1^{\pm 1}, \dots, a_m^{\pm 1}\}$, $m > 1$, $n \geq 2^{48}$ and n be divisible by 2^9 (from now on we impose these restrictions on m and n unless otherwise stated; note this estimate $n \geq 2^{48}$ has been improved on by Lysenok [L] to $n \geq 2^{13}$). By induction on i, let $B(m,n,0) = F_m$ and, assuming that the group $B(m,n,i-1)$ with $i \geq 1$ is already constructed as a quotient group of F_m, define A_i to be a shortest element of F_m (if any) the order of whose image in $B(m,n,i-1)$ (under the natural epimorphism $\psi_{i-1} : F_m \to B(m,n,i-1)$) is infinite. Then $B(m,n,i)$ is constructed as a quotient group of $B(m,n,i-1)$ by the normal closure of

$\psi_{i-1}(A_i^n)$. Clearly, $B(m,n,i)$ has a presentation of the form $B(m,n,i) = \langle\, a_1,\dots,a_m \parallel A_1^n,\dots,A_{i-1}^n, A_i^n \,\rangle$, where $A_1^n,\dots,A_{i-1}^n, A_i^n$ are the defining relators of $B(m,n,i)$. It is proven in [Iv] that for every i the word A_i does exist. Furthermore, it is shown that the direct limit $B(m,n,\infty)$ of the groups $B(m,n,i)$ as $i \to \infty$ (obtained by imposing on F_m of relators A_i^n for all $i = 1,2,\dots$) is exactly the free m-generator Burnside group $B(m,n)$ of exponent n ($B(m,n)$ is the quotient group F_n/F_m^n, where F_m^n is the subgroup generated by all nth powers of elements of F_m). The infiniteness of the group $B(m,n)$ already follows from the existence of the word A_i for every $i \geq 1$, since, otherwise, $B(m,n)$ could be given by finitely many relators and so A_i would fail to exist for sufficiently large i. It is also shown that the word and conjugacy problems for the group $B(m,n)$ are solvable (for odd $n \geq 665$ it is known due to Novikov and Adian [Ad]). As a matter of fact, these decision problems are effectively reduced to the word problem for groups $B(m,n,i)$ and it is shown that each $B(m,n,i)$ satisfies a linear isoperimetric inequality and hence $B(m,n,i)$ is a hyperbolic group in the sense of Gromov [G].

It is worth noting that the structure of finite subgroups of the groups $B(m,n,i)$, $B(m,n)$ is very complex when the exponent n is even and, in fact, finite subgroups of $B(m,n,i)$, $B(m,n)$ turn out to be so important in proofs of [Iv] that at least a third of article [Iv] is an investigation of their various properties and another third is a preparation of necessary techniques to conduct this investigation. The central result related to finite subgroups of the groups $B(m,n,i)$, $B(m,n)$ is the following: Let $n = n_1 n_2$, where n_1 is the maximal odd divisor of n. Then any finite subgroup G of $B(m,n,i)$, $B(m,n)$ is isomorphic to a subgroup of the direct product $D(2n_1) \times D(2n_2)^\ell$ for some ℓ, where $D(2k)$ denotes a dihedral group of order $2k$ (for odd $n \geq 665$ it is known due to Novikov and Adian that all finite and abelian subgroups of $B(m,n)$ are cyclic [Ad]). The natural question on whether this description is complete (i. e. every subgroup of $D(2n_1) \times D(2n_2)^\ell$ can actually be found in $B(m,n)$) is solved in the affirmative in a joint work with Ol'shanskii [IvOl1] by proving the following theorem: Let G be a finite 2-subgroup of $B(m,n)$. Then the centralizer $C_{B(m,n)}(G)$ of G in $B(m,n)$ contains a subgroup B isomorphic to a free Burnside group $B(\infty,n)$ of infinite countable rank such that $G \cap B = \{1\}$, whence $\langle G,\ B\rangle = G \times B$. (Since $B(\infty,n)$ obviously contains subgroups isomorphic to both $D(2n_1)$ and $D(2n_2)$ an embedding of $D(2n_1) \times D(2n_2)^\ell$ in $B(m,n)$ becomes trivial.) Among other results on subgroups of $B(m,n)$ proven in [IvOl1] are the following: The centralizer $C_{B(m,n)}(S)$ of a subgroup S is infinite if and only if S is a locally finite 2-group. Any infinite locally finite subgroup L is contained in a unique maximal locally finite subgroup while any finite 2-subgroup is contained in continuously many pairwise nonisomorphic maximal locally finite subgroups. A clear complete description of infinite (maximal) locally finite subgroups of $B(m,n)$ is also obtained.

The machinery developed in [Iv] for solving the Burnside problem for even exponents has turned out to be useful for other problems. For example, Gromov [G] conjectured that for every nonelementary (= non-cyclic-by-finite) hyperbolic group G there exists an $n = n(G)$ such that the quotient G/G^n is infinite. This Gromov conjecture (along with other results on such quotients G/G^n) is proven in [IvOl2] by carrying out the machinery of [Iv] to nonelementary hyperbolic groups.

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