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Abstracts of Plenary and Invited Lectures

Section:

2. Algebra

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Birational automorphisms of algebraic varieties

A *Fano fibration* is a projective algebraic variety V, equipped with a surjective morphism $\pi\colon V \to S$, the generic fiber F_η of which is a Fano variety over the (non-closed) field $\mathbf{C}(S)$, i. e. the anticanonical class $(-K_{F_\eta})$ is ample. Fano fibrations form the principal subject of birational geometry. A Fano fibration $\pi\colon V \to S$ is said to be *birationally rigid*, if for any birational map $\chi\colon V - - \to V'$ and any linear system $|D'|$ on V', free in codimension 1, there exists a fiber-wise birational automorphism $\chi^* \in \operatorname{Bir} F_\eta \subset \operatorname{Bir} V$ such that for $|D| = (\chi \circ \chi^*)_*^{-1}|D'|$ (the proper inverse image of $|D'|$ on V') the condition $|aD' + bK_{V'}| = \emptyset$ implies that $|aD + bK_V| = \emptyset$ for any $a, b \in \mathbf{Z}_+$. This property is very strong: it implies, for instance, that any birational map $\chi\colon V - - \to V'$ onto another Fano fibration is fiber-wise. In particular, birationally rigid V is non-rational and its group of birational automorphisms $\operatorname{Bir} V$ reduces to the group of birational automorphisms of the generic fiber, $\operatorname{Bir} F_\eta$. A Fano fibration is said to be *birationally superrigid*, if in the definition above, we can always take $\chi^* = \mathrm{id}_V$. For a superrigid Fano variety V with $\operatorname{Pic} V \cong \mathbf{Z}$ we have $\operatorname{Bir} V = \operatorname{Aut} V$. The first example of a birationally rigid threefold was given by V. A. Iskovskikh and Yu. I. Manin (smooth quartics in $\mathbf{P}^4$; 1971).
Recently, new series of birationally rigid Fano fibrations were discovered. Let $\pi\colon V \to \mathbf{P}^1$ be a smooth threefold, fibered into del Pezzo surfaces of degree $d = 1, 2$ or 3, $\operatorname{Pic} V = \mathbf{Z}K_V \oplus \mathbf{Z}F$, $F = \pi^{-1}(t)$ is a fiber, satisfying the following condition: $MK_V^2 - f$ is a non-effective 1-class for any $M \in \mathbf{Z}$, where $f \in A^2(V)$ is the class of a line in F.
Theorem 1. *For $d = 1$ the Fano fibration $\pi\colon V \to \mathbf{P}^1$ is birationally superrigid, for $d = 2, 3$ it is rigid.*
Let $V = V_d \subset \mathbf{P}^d$, $d \geq 5$, be a smooth hypersurface of degree d, satisfying the following condition: for any point $x \in V$ and any affine coordinate system with the origin at x the hypersurface V is given by the equation $f = f_1 + \cdots + f_d$, where the set of homogeneous polynomials $(f_1, \ldots, f_{d-1})$ make a regular sequence in $\mathcal{O}_{x,\mathbf{P}^d}$ (hypersurfaces satisfying this property form a non-empty Zariski open subset of the space of hypersurfaces of the given degree d).
Theorem 2. *The hypersurface V is birationally superrigid.*

Let $V = V_d$ be a sufficiently general quasi-smooth Fano hypersurface in the weighted projective space $\mathbf{P}^4(1, a_1, a_2, a_3, a_4)$, where $d = a_1 + a_2 + a_3 + a_4$. There are 95 families of these hypersurfaces.

Theorem 3. *V is birationally rigid.*

The last result is joint work with A. Corti and M. Reid.

Other examples of birationally rigid Fano fibrations will be discussed, together with some open problems and natural conjectures.

References: V. A. Iskovskikh, and Yu. I. Manin, Three-dimensional quartics and counterexamples to the Lüroth problem, Math. USSR Sbornik 86 (1971), 141-166.

A. V. Pukhlikov, Birational automorphisms of three-dimensional algebraic varieties with a pencil of Del Pezzo surfaces, Izvestiya 62 (1998), 123-164.