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Abstracts of Plenary and Invited Lectures

Section:

3. Number Theory and Arithmetic Algebraic Geometry

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P-adic analytic spaces

A valuation on a field k is a function $|\ |: k \to \mathbf{R}_+$ with the properties: $|x| = 0$ if and only if $x = 0$, $|xy| = |x| \cdot |y|$, and (the triangle axiom) $|x+y| \le |x| + |y|$. It is well known that either the completion of k with respect to a valuation is $\mathbf{R}$ or $\mathbf{C}$, the fields of real and of complex numbers, or the valuation on k satisfies the more strong (non-Archimedean) triangle axiom $|x+y| \le \max(|x|, |y|)$. The main (and the first discovered) example of a non-Archimedean valuation is the p-adic one on the field of rational numbers $\mathbf{Q}$. Here p is a prime number, and the p-adic valuation of a non-zero number of the form $p^n \frac{a}{b}$, where a and b are prime to p, is equal to $\frac{1}{p^n}$. The field of p-adic numbers $\mathbf{Q}_p$, the completion of $\mathbf{Q}$ with respect to the p-adic valuation, is of great importance in number theory. Is there geometry over $\mathbf{Q}_p$ as it is over $\mathbf{R}$ and $\mathbf{C}$? At the moment one is able to consider only analogs of complex analytic geometry since convergent power series make sense over $\mathbf{Q}_p$ as well, but analogs of functions of other types, e. g., smooth ones, are not yet known.

In comparison with $\mathbf{R}$ and $\mathbf{C}$, any field k complete with respect to a non-Archimedean valuation is totally disconnected and, therefore, the n-dimensional vector space k^n, considered with the functions which are locally convergent power series, is not an appropriate analog of the complex analytic space $\mathbf{C}^n$. In this connection, in rigid analytic geometry introduced by Tate in 1961 the analog of $\mathbf{C}^n$ is, roughly speaking, the space k^n provided with an extra structure. (More precisely, instead of k^n one considers the maximal spectrum of the ring of polynomials $k[T_1, \ldots, T_n]$, and the extra structure is a Grothendieck topology on it.) The framework of rigid analytic geometry enables one to construct a full analog of the complex analytic theory of coherent sheaves and their cohomology, but does not allow a direct application of the intuitive idea of continuity and, in particular, of the homotopy and singular homology notions.

It turns out that there exist p-adic analytic spaces to which the above notions can be directly applied. They are in a sense a geometrization of rigid analytic spaces, their definition is much simpler than that of the latter, and it is in fact a natural generalization of that of complex analytic spaces. For example, the analog of $\mathbf{C}^n$ is the space $\mathbf{A}^n$ of the multiplicative semi-norms on the ring $k[T_1, \ldots, T_n]$ that extend the valuation on k, and the analytic functions on it are local limits of rational functions. (The vector space k^n

is naturally embedded in $\mathbf{A}^n$, and the topology of $\mathbf{A}^n$ induces the canonical topology on k^n.) Basic topological properties of the analytic spaces are as follows. Each point has a fundamental system of open neighborhoods which are locally compact, countable at infinity and arcwise connected, and the topological dimension of a paracompact analytic space is equal to its (algebraic) dimension.

Furthermore, recall that a formal scheme over the ring of integers k° of k has the closed fibre $\mathcal{X}_s$, which is a scheme of locally finite type over the residue field of k, and the generic fibre $\mathcal{X}_\eta$, which is an analytic space over k. It turns out that, if the canonical morphism from $\mathcal{X}$ to the formal spectrum of k° is a composition of semi-stable morphisms, then the homotopy type of $\mathcal{X}_\eta$ (and of any dense Zariski open subset of $\mathcal{X}_\eta$) coincides with that of the geometric realization of a simplicial set associated to $\mathcal{X}_s$. (For example, the analytification of a Zariski open subset of a proper scheme with good reduction is contractible, and the Tate elliptic curve is homotopy equivalent to the circle.) The above fact and results of J. de Jong on alterations are used to prove that any smooth analytic space over k is locally contractible. In particular, such a space has a universal covering which is a Galois one with the Galois group isomorphic to the fundamental group of the space and, if the space is paracompact , its cohomology groups with coefficients in a constant sheaf coincide with the singular cohomology groups. One also proves that, if X is the analytification of a proper scheme over k, then the cohomology groups $H^q(X, \mathbf{Z})$ are finitely generated, and there exists a finite separable extension k' of k such that for any non-Archimedean field K over k' one has $H^q(X\widehat{\otimes}k', \mathbf{Z}) \xrightarrow{\sim} H^q(X\widehat{\otimes}K, \mathbf{Z})$.

As it is clear from the above, the (topological) cohomology groups of analytic spaces over k don't play the same role as the similar cohomology groups of complex analytic spaces. The role of the latter is played by étale cohomology groups. Étale cohomology theory developed is parallel to that for schemes but, in comparison with the latter, is much simpler, and the reason for this is that the analytic spaces have good topology. Étale cohomology of the analytic spaces is used for the study of the vanishing cycles sheaves of schemes and for constructing smooth representations of p-adic reductive groups.