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3. Number Theory and Arithmetic Algebraic Geometry

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Colmez, Pierre, CNRS, École Normale Supérieure et Institut de Math. de Jussieu, Paris, France
(φ, Γ)-modules and Iwasawa theory of p-adic representations

The talk will try to report on the classification of p-adic representions of a local field in terms of (φ, Γ)-modules. This classification was introduced by Fontaine some ten years ago and brings a new perspective on local class field theory and its generalisations. It is still in progress but a number of interesting results (due to various people) have already been obtained especially in the applications to Iwasawa theory. Before describing a little bit more in detail what the content of the talk should be, let us introduce some notation.

Fix an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$. If K is a finite extension of $\mathbf{Q}_p$ and $n \in \mathbf{N}$, set $K_n = \mathbf{Q}_p(\mu_{p^n})$ and let K_∞ be the cyclotomic extension of K union of all the K_n. Let $\mathcal{G}_K$ be the Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}_p/K)$ and $\mathcal{H}_K = \mathrm{Gal}(\overline{\mathbf{Q}}_p/K_\infty)$. Finally, set $\Gamma_K = \mathcal{G}_K/\mathcal{H}_K = \mathrm{Gal}(K_\infty/K)$.

The theory of (φ, Γ_K)-modules attaches to any p-adic representation V of $\mathcal{H}_K$ a module $D(V)$ over a ring of power series in one variable, endowed with an action of the Frobenius φ. The main point of the theory is that one can recover V from $D(V)$ which is a priori much simpler (Γ_K being in general procyclic, $D(V)$ is given by two operators (φ and a generator of Γ_K) satisfying a commutation relation). The theory of p-adic representations of $\mathcal{G}_K$ is thus equivalent to the study of (φ, Γ)-modules which naturally gives rise to three types of questions.
– Can one read the properties of the representation directly on its (φ, Γ)-module?
– Can one recover from $D(V)$ other classical invariants of V and use $D(V)$ to attach new invariants to V?
– Can one use (φ, Γ)-modules to construct interesting p-adic representations of $\mathcal{G}_K$?

Concerning the first question, the idea is that, the nicer the representation is, the nicer its (φ, Γ)-module must be. In this direction one can prove that representations of $\mathcal{G}_K$ are nicer than general representations of $\mathcal{H}_K$: they are *overconvergent*. If K is unramified over $\mathbf{Q}_p$, crystalline representations of $\mathcal{G}_K$ are as nice as possible: they are *of finite height*.

Concerning the second question, the most interesting result is a description of the Galois cohomology of V in terms of $D(V)$ from which one can recover

Tate's local duality and describe the Iwasawa modules attached to V. One obtains in this way a vast generalisation of Coleman's isomorphism and, if the representation is assumed to be de Rham, of Coates-Wiles homomorphisms as well as an explicit reciprocity law relating these Coates-Wiles homomorphisms to Bloch-Kato exponential maps (these last results use the overconvergence of representations of $\mathcal{G}_K$). It is not unreasonable to expect that these results will share some new light on p-adic L-functions. One can also, using the infinitesimal action of Γ_K, attach to V an overconvergent differential module endowed with a Frobenius and use this module to recover the generalised Hodge-Tate decomposition of Sen and characterise de Rham representations.

Finally, one can construct (φ, Γ)-modules having good properties and recover the results of Fontaine and Laffaille on crystalline representations from this construction.