ICM 1998, Berlin, Aug. 18–27

Abstracts of Plenary and Invited Lectures

Section:

3. Number Theory and Arithmetic Algebraic Geometry

1991 MS Classification: 11
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Bounds for arithmetic multiplicities

An important role in several areas of Number Theory is played by multiplicities, those of automorphic representations and of zeros of L-functions being prime examples. While one would ultimately like to understand such multiplicities exactly, in many cases we only have conjectures and, concerning estimates for sizes, only trivial ones. This is particularly true when the multiplicity cannot be effectively isolated using standard analytic tools such as the trace formula or the explicit formula for the zeros of L-functions. In some cases there has been much work constructing explicit examples and hence lower bounds for multiplicities. Upper bounds also present a serious challenge. The work of Phillips and Sarnak and later of Wolpert on the disappearance of cusp forms illustrates this in the case of the spectral multiplicity of congruence groups.

This lecture will describe some recent applications of techniques that allow one to give non-trivial upper bounds for multiplicities in certain arithmetic instances. These applications include in particular estimates for dimensions of spaces of cusp forms of weight one, sizes of bounded order subgroups of class groups, multiplicities of number fields with a given degree and discriminant and numbers of elliptic curves with various arithmetic conditions. The techniques share a common strategy, which combines spectral mean value theorems with special constraints only present in arithmetic cases. In the example of bounding the dimensions of spaces of cusp forms of weight one, this combination of ideas uses the Hilbert space structure of automorphic forms together with the rigidity of Galois representations, whose existence is guaranteed by the Deligne-Serre theorem.

In addition to giving particular applications we shall suggest conjectures and offer some general directions for further work on the estimation of arithmetic multiplicities.