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Section:

3. Number Theory and Arithmetic Algebraic Geometry

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Some results of algebraic independence

The most impressive result of the last years is due to Yu. V. Nesterenko in 1996 [N2] and concerns modular functions.
With Ramanujan's notation, let P, Q, R be the three first Eisenstein series $P(z) = 1 - 24\sum_{n\geq 1}\sigma_1(n)\, z^n$, $Q(z) = 1 + 240\sum_{n\geq 1}\sigma_3(n)\, z^n$, and $R(z) = 1 - 504\sum_{n\geq 1}\sigma_5(n)\, z^n$, where, for $n \in \mathbf{N}$, $\sigma_k(n) = \sum_{d|n} d^k$. The fundamental result is the following:
Theorem (Nesterenko). *If $q \in \mathbf{C}$ satisfies $0 < |q| < 1$, then the transcendence degree over $\mathbf{Q}$ of the field $\mathbf{Q}(q, P(q), Q(q), R(q))$ is at least 3.*
The links between Eisenstein series and elliptic curves or elliptic functions give, for example the following corollaries:
Corollary 1. *Let $\wp$ be an elliptic Weierstrass function with lattice of periods $\mathbf{Z}\omega_1 + \mathbf{Z}\omega_2$, where $\omega_2/\omega_1 = \tau$. Let η_1 be the quasi-period associated to ω_1 and suppose that the invariants g_2 and g_3 are algebraic. Then the three numbers $e^{2i\pi\tau}$, ω_1/π and η_1/π are algebraically independent over $\mathbf{Q}$.*
Corollary 2. *With the same notation, in the case of complex multiplication, i. e. if τ is an algebraic number of degree 2, and if η is the quasi-period associated to the period $\omega \neq 0$ of $\wp$, then each set $\{e^{2i\pi\tau}, \omega, \eta\}$ and $\{e^{2i\pi\tau}, \omega, \pi\}$ consists of 3 algebraically independent (over $\mathbf{Q}$) numbers.*
In particular, for $\tau = i$, with the elliptic curve of equation $y^2 = 4x^3 - 4x$, one gets **the algebraic independence of the three numbers** π, e^π and $\Gamma(1/4)$.
This is the first result of algebraic independence about values of modular functions obtained by the use of modularity properties and not by elliptic properties. In the 70's, G. V. Chudnovsky obtained transcendence degree 2 by elliptic methods, for example the algebraic independence of ω_1/π and η_1/π under the hypothesis of corollary 1, and of ω and π under the hypothesis of corollary 2, in particular the algebraic independence of π and $\Gamma(1/4)$.
Another translation of the theorem in terms of Jacobi theta-functions gives, for example, the transcendence of $\sum_{n\geq 1} \ell^{-n^2}$ for any positive integer $\ell \geq 2$. This number was considered by Liouville (1844) who proved its irrationality.
Nesterenko's proof of the theorem has a classical form, using the construction (by the box principle) of an auxiliary function, a zero estimate and a criterion for algebraic independence (see the talk [N1] of Nesterenko at

ICM90). Notice that this proof gives also quantitative results (measures of algebraic independence).

But, in the recent years, appeared new proofs of algebraic independence, which proceed from another approach. The new methods developped by M. Waldschmidt and D. Roy ([RW1] and [RW2]), M. Laurent and D. Roy ([LR]) and P. Philippon ([P1] and [P2], which present a systematic treatment for transcendence and algebraic independence) use different points of vue, but the three have two common points: they use interpolation determinants (introduced in the 80's by M. Laurent in transcendence proofs) instead of auxiliary functions and they use general properties of Diophantine approximation.

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