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Section:

3. Number Theory and Arithmetic Algebraic Geometry

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The Intrinsic Hodge Theory of p-adic Hyperbolic Curves

A *hyperbolic curve* is an algebraic curve obtained by removing r points from a smooth, proper curve of genus g, where g and r are nonnegative integers such that $2g-2+r>0$. If X is a hyperbolic curve over the field of complex numbers $\mathbf{C}$, then X gives rise in a natural way to a Riemann surface $\mathcal{X}$. As one knows from complex analysis, the most fundamental fact concerning such a Riemann surface (due to Köbe) is that it may be *uniformized by the upper half-plane*, i. e., $\mathcal{X} \cong \mathfrak{H}/\Gamma$ where $\mathfrak{H} \stackrel{\text{def}}{=} \{z \in \mathbf{C} \mid \text{Im}(z) > 0\}$, and $\Gamma \cong \pi_1(\mathcal{X})$ (the topological fundamental group of $\mathcal{X}$) is a discontinuous group acting on $\mathfrak{H}$. Note that the action of Γ on $\mathfrak{H}$ defines a canonical representation $\rho_{\mathcal{X}} : \pi_1(\mathcal{X}) \to PSL_2(\mathbf{R}) = \text{Aut}_{\text{Holomorphic}}(\mathfrak{H})$. Now there are (at least) two ways to interpret the Köbe uniformization:
The Physical Approach: The Köbe uniformization provides a "physical, " geometric way to reconstruct the algebraic curve X from its topological fundamental group $\pi_1(\mathcal{X})$, equipped with some canonical *arithmetic* information, i. e. , $\rho_{\mathcal{X}}$.
The Modular Approach: The representation $\rho_{\mathcal{X}}$ gives rise to a special kind of $\mathbf{P}^1$-bundle with connection $(P \to X, \nabla_P)$ on the algebraic curve X called an *indigenous bundle*. More generally, the moduli space of indigenous bundles forms an algebraic torsor $\mathcal{S}_{g,r} \to \mathcal{M}_{g,r}$ (in fact, defined over $\mathbf{Z}[\frac{1}{2}]$), called the *Schwarz torsor*, over the moduli stack of hyperbolic curves of type (g,r). Then Teichmüller theory (i. e., in particular, the uniformization theory of hyperbolic curves and their moduli) may be formulated as the study of the *canonical real analytic section of the Schwarz torsor* defined by the indigenous bundle arising from the Köbe uniformization.
The aim of the present talk is to discuss various generalizations to the p-adic case of these physical and modular aspects of Köbe's uniformization theorem.
Note that both the physical and modular approaches to the Köbe uniformization assert that there is a certain equivalence

$$\text{algebraic geometry} \Longleftrightarrow \text{topology endowed with an arithmetic structure}$$

That is, on the algebraic geometry side, we have the scheme (respectively, stack) given by the curve X itself in the physical approach (respectively, its moduli $\mathcal{M}_{g,r}$ in the modular approach), whereas on the "topology plus arithmetic structure" side, we have the theory of the canonical representation $\rho_{\mathcal{X}}$ of $\pi_1(\mathcal{X})$. This sort of equivalence is reminiscent of that given by classical or p-adic *Hodge theory* between the de Rham or Hodge cohomology of an algebraic variety (on the algebraic geometry side), and the singular or étale cohomology (equipped with Galois action) on the topology plus arithmetic side. In our case, however, instead of dealing with the cohomology of the curve, we are dealing with "the curve itself" and its moduli. It is for this reason that we refer to this theory as the *intrinsic Hodge theory* of the curve X.

Now let X be a hyperbolic curve over a finite extension K of the field $\mathbf{Q}_p$ of p-adic numbers. Let $\pi_1(X)$ be its *algebraic fundamental group*. Note that $\pi_1(X)$ comes equipped with a natural augmentation to Γ_K, the absolute Galois group of K. Then one has the following p-adic analogue (cf. [4]) of the *physical aspect* of the Köbe uniformization: *One may recover the scheme X (in the sense of the functor defined by X by considering dominant K-morphisms to X from K-smooth schemes S of finite type) as the functor of open, continuous Γ_K-compatible homomorphisms $\pi_1(S) \to \pi(X)$.* This sort of result was predicted by Grothendieck's *anabelian philosophy*, and builds on earlier work of H. Nakamura and A. Tamagawa. It may also be considered as the analogue for hyperbolic curves of the Tate conjecture on homomorphisms between abelian varieties.

Finally, we have a p-adic analogue of the *modular aspect* of the Köbe uniformization, i. e., of Teichmüller theory. Just as Teichmüller theory may be regarded as the theory of the canonical real analytic section of the Schwarz torsor, the p-adic analogue of Teichmüller theory concerns the issue of finding *canonical p-adic sections of the Schwarz torsor.* In characteristic p, this is achieved by considering the stack $\mathcal{N}_{g,r}$ of indigenous bundles with square nilpotent p-curvature. In our talk, we will discuss work on the fine structure of $\mathcal{N}_{g,r}$ (cf. [3]) in a "neighborhood of infinity" (i. e., near totally degenerate curves), as well as the theory of canonical p-adic liftings (cf. [1,2]) for various portions of $\mathcal{N}_{g,r}$.

Bibliography

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