# ICM 1998, Berlin, Aug. 18–27

## Abstracts of Plenary and Invited Lectures

### Section:

## 3. Number Theory and Arithmetic Algebraic Geometry

1991 MS Classification: 11, 14
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**Small points, Heegner points, and Arakelov theory**

In this talk, I will explain the recent applications of Arakelov theory to the Bogomolov conjecture on small points, and to the Gross-Zagier formula for Heegner points.

**Theorem A (Bogomolov conjecture).** *Let $Y$ be a subvariety of an abelian variety $A$ defined over $\bar{\mathbb{Q}}$. Let $h : A(\bar{\mathbb{Q}}) \to \mathbb{R}$ be a Néron-Tate height function (with respect to a symmetric projective embedding of $A$). Assume that $Y$ is not a translation of an abelian subvariety by a torsion point. Then there is a positive number $\epsilon$ such that the set*

$$\{x \in Y(\bar{\mathbb{Q}}) : \quad h(x) < \epsilon\}$$

*is not Zariski dense in $Y$.*

**Remarks**:
1. As the above set contains all torsion points in $Y$, the above theorem implies a theorem of Raynaud on Lang's conjecture.
2. When $Y$ is the image of a map $X \to A = \mathrm{Jac}(X)$ of a curve to its Jacobian defined by the canonical divisor of $X$, this theorem is due to E. Ullmo.

Actually one can prove the following stronger statement about probability measure rather than Zariski topology:

**Theorem B.** *Let $(x_n, n \in \mathbb{N})$ be a sequence of points in $A(\bar{\mathbb{Q}})$ such that the following conditions are verified:*
*1. There is no subsequence of $(x_n, n \in \mathbb{N})$ contained in a translation of proper abelian subvariety by a torsion point;*
*2.* $\lim_{n\to\infty} h(x_n) = 0$.
*Then the probability measures of the Galois orbits of $x_n$ converge to the Haar measure of $A(\mathbb{C})$.*

One can use this to show that the set of torsion points on $A$ over the maximal totally real fields is finite.

**Theorem C (Gross-Zagier formula).** *Let $E$ be a modular elliptic curve defined over a totally real field $F$ with the conductor $\mathfrak{n}$. Let $K/F$ be an imaginary quadratic extension which is unramified over places dividing $\mathfrak{n}$*

*and such that the functional equation of $L(E/K, s)$ has the negative sign. Then*

$$L'(E/K, 1) = \text{(positive constant)} \cdot h(y_K)$$

*where $y_K \in E(K)$ is the Heegner point, and $h$ is the Néron-Tate height on $E$.*

**Remarks:**

1. Here we say $E/F$ is modular if $L(E, s) = L(f, s)$ with $f$ a Hilbert modular form over $F$. In our case, $E$ has a uniformization by certain Shimura curve (by the Jacquet-Langlands theory and the Eisenstein divisor class). The points $y_K$ is the image of certain CM-points on Shimura curves.
2. When $F = \mathbb{Q}$ and every factor of $\mathfrak{n}$ is split in $K$, this theorem is due to Gross and Zagier. Modulo the construction of Eisenstein divisor class, the general statement was first conjectured by Gross.

Assume that $\mathrm{ord}_{s=1}L(E, s) \leq 1$, and that either $g$ is odd or $\mathfrak{n}$ is not a square. Then one may show that there is a $K$ as in the theorem such that $L'(E/K, 1) \neq 0$. This implies that $y_K$ is nontorsion. By Kolyvagin's theory of Euler system, this in turn implies the following statement concerning the Birch and Swinnerton-Dyer conjecture:

**Theorem D**. *Assume that $\mathrm{ord}_{s=1}L(E, s) \leq 1$, and that either $g$ is odd or $\mathfrak{n}$ is not a square. Then*

1. *The Mordell-Weil group $E(F)$ has the rank equal to $\mathrm{ord}_{s=1}L(E, s)$.*
2. *The Shafarevich group of $E$ over $F$ is finite.*

In the proofs of both Theorem A and Theorem C, Arakelov theory (developed by Deligne, Faltings, Gillet, Soulé, Szpiro, etc.) plays the crucial rule in the computation of Néron-Tate heights. By Hodge index theorem in this theory, one can either express the Néron-Tate heights as the intersections of arithmetic line bundles on the integral models of abelian varieties, or on the integral models (i. e., arithmetic surfaces) of curves.