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Section:

4. Algebraic Geometry

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Green, Mark, University of California, Los Angeles, California (USA)
Higher Abel-Jacobi Maps

For a smooth projective variety X, the structure of the Chow group $CH^p(X)$ representing codimension p algebraic cycles modulo rational equivalence, is still basically a mystery when $p > 1$. There is a conjectural filtration proposed by Bloch-Beilinson-Deligne-Murre $F^m CH^p(X) \otimes_{\mathbf{Z}} \mathbf{Q}$ whose graded pieces ought to have higher Abel-Jacobi maps $\mathrm{Gr}^m CH^p(X) \otimes_{\mathbf{Z}} \mathbf{Q} \to J^p_m(X)$. The cases $0 \leq p, m \leq 1$ are known (divisors, Hodge class, Abel-Jacobi map). The case $p = 2$, $m = 2$ (kernel of the Albanese map for 0-cycles on surfaces) is an interesting first case to consider. A "toy model" for this is the relative Chow group $CH^2(\mathbf{P}^2, T)$, where $T \subset \mathbf{P}^2$ is the triangle $z_0 z_1 z_2 = 0$, roughly described as 0-cycles on $\mathbf{P}^2 - T = \mathbf{C}^* \times \mathbf{C}^*$, modulo divisors of meromorphic functions f on curves $C \subset \mathbf{P}^2$ such that $f = 1$ on $C \cap T$. There is a series of maps

$$\psi_0 \colon CH^2(\mathbf{P}^2, T) \to \mathbf{Z};$$
$$\psi_1 \colon \ker(\psi_0) \to \mathbf{C}^* \oplus \mathbf{C}^*;$$
$$\psi_2 \colon \ker(\psi_1) \to K_2(\mathbf{C}).$$

It is known (Bloch, Suslin) that these are all surjective and ψ_2 is injective. These all have simple algebraic descriptions—ψ_0 is degree; $\psi_1(a, b) = a \oplus b$; $\psi_2(a, b) = \{a, b\}$. The essential tool in proving this is the Suslin reciprocity theorem.

This example provides a model for the general case—one should think of $(\mathbf{P}^2, T)$ as analogous to a complete surface with $h^{2,0} = 1$. $K_2(\mathbf{C})$ is infinite-dimensional, as Mumford's theorem asserts for surfaces with $h^{2,0} \neq 0$. For a general X, one has

$$\psi_0 \colon CH^2(X) \to \mathrm{Hdg}^2(X);$$
$$\psi_1 \colon \ker(\psi_1) \to J^2(X);$$

where ψ_0 is the cycle class map to the Hodge classes on X, and ψ_1 is the Abel-Jacobi map. We have constructed part of the missing map ψ_2 in the case of 0-cycles on a surface.

The regulator map for a curve

$$X - D \xrightarrow{(f,g)} \mathbf{C}^* \times \mathbf{C}^*$$

is a homomorphism $r\colon \pi_1(X-D) \to \mathbf{C}/\mathbf{Z}(2)$ given by

$$r(\gamma) = \int_\gamma \log(f)\frac{dg}{g} \log(g(p)) \int_\gamma \frac{df}{f},$$

where p is a base-point on γ; the answer does not depend on p. If $\gamma = \partial U$ for U a disc in $\mathbf{C}^* \times \mathbf{C}^*$, then

$$r(\gamma) = \int_U \frac{df}{f} \wedge \frac{dg}{g}.$$

This formula generalizes to a definition in the more general situation of a non-singular curve C and a map $f\colon C \to X$ to a smooth projective surface X. If $\mu \in \ker(H^2(X,\mathbf{Z}) \xrightarrow{f^*} H^2(C,\mathbf{Z}))$, then representing μ by a real d and d^c-closed 2-form, $f^*\mu = dd^c g$ for $g \in A^0(C)$, unique up to adding a constant. If $\gamma \in \ker(H_1(C,\mathbf{Z}) \to H_1(X,\mathbf{Z}))$, so that $\gamma = \partial\Gamma$ in X, then we define

$$e_{X,C}(\mu,\gamma) = \int_\Gamma \mu - \int_\gamma f^*(d^c g) \in \mathbf{R}/\mathbf{Z},$$

which does not depend on any of the choices. These quantities are known as **membrane integrals**. More intrinsically, $e_{X,C}$ is the extension class of the extension of mixed Hodge structures (MHS)

$$0 \to \operatorname{coker}(H^1(X) \to H^1(C)) \to H^2(X,C) \to \ker(H^2(X) \to H^2(C)) \to 0.$$

Denote the term on the left $H^1(C)_{\text{new}}$ and the term on the right $H^2(X)_C$; now by standard identifications

$$e_{X,C} \in (\mathbf{R}/\mathbf{Z}) \otimes_{\mathbf{Z}} \operatorname{Hom}_{\mathbf{Z}}(H^2(X)_C, H^1(C)_{\text{new}}).$$

The basic idea is that $e_{X,C}$ should describe the map $f_*\colon J(C)_{\text{new}} \to CH^2(X)$. If Z is a 0-cycle on X in the kernel of the Albanese map, then if we choose C so that Z lies on $f(C)$ and lifts to $\tilde{Z} \in Z^1(C)$ such that $f_*\tilde{Z} = Z$ and $\deg(Z) = 0$ on each component of C, then $AJ_C(\tilde{Z})$ is represented by the extension class $e_{C,\tilde{Z}}$ of the extension of mixed Hodge structures of $H^1(C)_{\text{new}}$ by 1, and by standard identifications

$$e_{C,\tilde{Z}} \in (\mathbf{R}/\mathbf{Z}) \otimes_{\mathbf{Z}} \operatorname{Hom}_{\mathbf{Z}}(H^1(C)_{\text{new}}, 1).$$

The two extensions of MHS fit together to give a 2-step extension of MHS of $H^2(X)_C$ by 1, which unfortunately cannot be used directly, since Ext^2's all are 0 for MHS. We combine the transcendental part of $e_{X,C}$ with $e_{C,\tilde{Z}}$. The tensor product followed by contraction gives an element

$$e_{X,C,\tilde{Z}} = e_{C,\tilde{Z}} \circ (e_{X,C})_{\text{tr}} \in (\mathbf{R}/\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{R}/\mathbf{Z}) \otimes_{\mathbf{Z}} \operatorname{Hom}_{\mathbf{Z}}(H^2(X)_{\text{tr}}, 1).$$

If we let $U_2^2(X) = \{e_{X,C,\tilde{Z}} \mid f_*\tilde{Z} = 0 \text{ as a } 0\text{−cycle on } X\}$ and

$$J_2^2(X) = \frac{(\mathbf{R}/\mathbf{Z}) \otimes_{\mathbf{Z}} (\mathbf{R}/\mathbf{Z}) \otimes_{\mathbf{Z}} \mathrm{Hom}_{\mathbf{Z}}(H^2(X)_{\mathrm{tr}}, 1)}{U_2^2(X)},$$

then $Z \mapsto [e_{X,C,\tilde{Z}}]$ gives a well-defined invariant

$$\psi_2 \colon \ker(\psi_1) \to J_2^2(X)$$

that is independent of the choices of C and $\tilde{Z}$, and which depends only on the rational equivalence class of Z on X. It is necessary to allow reducible curves C. Claire Voisin has shown that for surfaces with $h^{2,0} \neq 0$, the map ψ_2 has infinite-dimensional image, and also that it need not be injective.
An invariant which complements the one above was obtained in joint work with Phillip Griffiths. By work of Katz and Grothendieck, there is for any smooth projective variety X defined over $\mathbf{C}$ the arithmetic Gauss-Manin connection

$$\nabla_{X/\mathbf{Q}} \colon H^k(X, \mathbf{C}) \to \Omega^1_{\mathbf{C}/\mathbf{Q}} \otimes_{\mathbf{C}} H^k(X, \mathbf{C}).$$

We define an **arithmetic Hodge structure (AHS)** to be a complex vector space V with a finite descending filtration $F^\bullet V$ and a connection $\nabla \colon V \to \Omega^1_{\mathbf{C}/\mathbf{Q}} \otimes_{\mathbf{C}} V$ satisfying $\nabla^2 = 0$ and $\nabla F^p V \subseteq \Omega^1_{\mathbf{C}/\mathbf{Q}} \otimes_{\mathbf{C}} F^{p-1}V$ (Griffiths transversality) for all p. A short exact sequence $0 \to A \to B \to C \to 0$ of AHS has extension class

$$e \in \mathrm{Ext}^1_{\mathrm{AHS}}(C, A) = H^1(\Omega^\bullet_{\mathbf{C}/\mathbf{Q}} \otimes_{\mathbf{C}} F^{-\bullet}\mathrm{Hom}_{\mathbf{C}}(C, A), \nabla_{\mathrm{Hom}_{\mathbf{C}}(C,A)}).$$

A 2-step exact sequence $0 \to A \to B \to C \to D \to 0$ of AHS has a well-defined map from the Yoneda Ext

$$\mathrm{Ext}^2_{\mathrm{AHS}}(D, A) \to H^2(\Omega^\bullet_{\mathbf{C}/\mathbf{Q}} \otimes_{\mathbf{C}} F^{-\bullet}\mathrm{Hom}_{\mathbf{C}}(D, A), \nabla_{\mathrm{Hom}_{\mathbf{C}}(D,A)})$$

obtained by composing the extension class of the two 1-step extensions $0 \to A \to B \to E \to 0$, $0 \to E \to C \to D \to 0$ it breaks into. This is exactly the obstruction to finding an AHS V with B as a sub-AHS and C as a quotient AHS in a compatible way. In contrast to MHS, AHS have non-trivial Ext^2's, and these have geometric meaning.
This extension class theory fits in well with the pre-existing **arithmetic cycle class map**

$$\eta \colon CH^p(X) \otimes_{\mathbf{Z}} \mathbf{Q} \to \mathbf{H}^{2p}(\Omega^{\geq p}_{X/\mathbf{Q}})$$

whose graded pieces are $\eta_m \colon \ker(\eta_{m-1}) \to H^m(\Omega^\bullet_{\mathbf{C}/\mathbf{Q}} \otimes_{\mathbf{C}} F^{p-\bullet}H^{2p-m}(X, \mathbf{C}), \nabla_{X/\mathbf{Q}})$; one expects these to be consistent with the conjectural Bloch-Beilinson-Deligne-Murre filtration on $CH^p(X) \otimes_{\mathbf{Z}} \mathbf{Q}$. For 0-cycles on a surface, we are able to show that η_2 is the element of $\mathrm{Ext}^2_{\mathrm{AHS}}(H^2(X), 1)$ coming from the 2-step extension of AHS using $Z \subset C \subset X$ analogous to the construction above in the mixed Hodge structure case.