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Abstracts of Plenary and Invited Lectures

Section:

4. Algebraic Geometry

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Operads and algebraic geometry

The study (motivated by mathematical physics) of algebraic varieties related to the moduli spaces of curves, helped to uncover important connections with the abstract algebraic theory of operads. This interaction led to new developments in both theories, and the purpose of the talk is to discuss some of them.

1. Operads. To explain why operads are interesting, it may be useful to recall a well known problem of Hilbert (as it came to be understood later). Let P be some class of functions considered in analysis (e. g. , continuous, smooth, algebraic etc.). Is it possible to express any function $f : \mathbf{R}^n \to \mathbf{R}$ of class P, depending of $n \geq 3$ variables, as a superposition of functions of class P, depending on 1 or 2 variables only?
For a set X, maps $X^n \to X$ are also called n-ary operations on X. A collection P of such maps (with possibly varying n) is called an *operad*, if it is closed under arbitrary superpositions as well as under permutations of variables. Thus $P(n)$, the n-ary part of P, is acted upon by the symmetric group S_n. One can view P as a multivariable analog of a (semi)group of transformations of X.
As with groups, the actual working definition (due to J. P. May) splits the above naive one into two. An abstract operad is a collection $P = \{P(n), n \geq 0\}$ of sets equipped with S_n-actions and with certain maps mimicking the functional suporpositions. Then, an action of P on a set X is a rule which associated to any $\mu \in P(n)$ an n-ary operation in a compatible way. Instead of sets one can consider vector spaces (resp. cochain complexes), requiring all multi-variable maps to be multilinear. In this way we get linear (resp. $dg-$) operads.

2. Main examples. First, many familiar algebraic structures can be described via operad actions. For example, we have linear operads $\mathcal{A}s$, $\mathcal{C}om$, $\mathcal{L}ie$ governing associative, commutative and Lie algebras.
Second, the theory of moduli spaces provides examples of operads in the category of algebraic varieties or topological spaces. The most important are $\overline{\mathcal{M}}_0 = \{\overline{\mathcal{M}}_0(n)\}$, where $\overline{\mathcal{M}}_0(n)$ is the moduli space of stable $n+1$-pointed curves of genus 0, and $\mathcal{M}_0 = \{\mathcal{M}_0(n)\}$ where $\mathcal{M}_0(n)$ is the moduli space of

spheres with $n+1$ parametrized holes. One can consider curves/Riemann surfaces of arbitrary genus, thus obtaining operads $\overline{\mathcal{M}}, \mathcal{M}$. By taking the (topological) homology of the above moduli spaces, one gets graded linear operads $H_\bullet(\overline{\mathcal{M}}_0)$ etc. The theory of Gromov-Witten invariants [6] defines an action of $H_\bullet(\overline{\mathcal{M}}_0)$ on the space $H^\bullet(V, \mathbf{C})$ for any smooth projective variety V over $\mathbf{C}$.

Third, the graph complexes of Kontsevich (and their analogs) lead to dg-operads relating the first two groups of examples.

3. Operad duality. To get a conceptual understanding of the (tree part of the) graph complexes, V. Ginzburg and the author [3] introduced a certain duality among linear (and more generally, *dg*-) operads.

A linear operad P is called quadratic, if every $P(n)$ is spanned by superposition of elements of $P(2)$ and also all the relations among elements of $P(2)$ follow from those holding already in $P(3)$. For such P, its Koszul dual $P^!$ is defined to have the dual space of generators and orthogonal space of relations. For example, $\mathcal{C}om^! = \mathcal{L}ie$ while $\mathcal{A}s^! = \mathcal{A}s$. More generally, for any dg-operad P a version of the graph complex construction gives the cobar-dual operad $D(P)$. For a quadratic P there is a natural map $D(P) \to P^!$, and P is called Koszul if it is a quasi-isomorphism. For example, $\mathcal{A}s, \mathcal{C}om, \mathcal{L}ie$ are Koszul. E. Getzler generalized the concept of Koszul duality to operads with generators in any degree and proved that $H_\bullet(\overline{\mathcal{M}}_0)$ is Koszul.

4. Cyclic and modular operads. Operads coming from moduli spaces possess some extra structure. This structure was axiomatized by E. Getzler and the author [1-2] in the concepts of cyclic and modular operads. Cyclic operads posess the action of S_{n+1}, not just S_n, on the space of n-ary operations. This structure is present on the operads related to genus 0 moduli spaces. Modular operads are, in addition, graded by an extra parameter ("genus") and possess contraction maps lowering the arity and raising the genus (with moduli spaces of curves and their homology giving prime examples). The cobar-duality can be generalized to cyclic and modular cases. The generating function of the dual operad is for ordinary operads given by the composition inverse, for cyclic operads by the Legendre transform and for modular operads by the Fourier transform (in an infinite-dimensional setting).

5. Geometry based on an operad. An affine algebraic variety can be described in terms of its commutative algebra of functions. So if P is a linear operad related to $\mathcal{C}om$ by a morphism of operads, it becomes possible to speak about (semiformal) P-manifolds as ordinary smooth algebraic varieties equipped with some extra structure reflecting P. For $P = \mathcal{A}s$ the theory of such objects ("noncommutative manifolds") was worked out in [5]. The general feature of such P-geometry is that the curvature data satisfy the relations of the dual operad $P^!$. For the case $P = \mathcal{C}om$ (i. e., for ordinary manifolds) this means that the Atiyah class of any complex manifold satisfies the Jacobi identity [4].

[1] E. Getzler, M. Kapranov, Cyclic operads and cyclic homology, in "Geometry, Topology and Physics for R. Bott" (S. T. Yau, Ed.) p.167-201, Intern. Press, Cambridge MA, 1995.

[2] E. Getzler, M. Kapranov, Modular operads, *Compositio Math.* **110** (1998), 65-126.

[3] V. Ginzburg, M. Kapranov, Koszul duality for operads, *Duke Math. J.* **76** (1994), 203-272.

[4] M. Kapranov, Rozansky-Witten invariants and Atiyah classes, preprint alg-geom/9704009, *Compositio Math.* , to appear.

[5] M. Kapranov, Noncommutative geometry based on commutator expansions, preprint math. AG/9802041.

[6] M. Kontsevich, Y. Manin, Gromov-Witten classes, quantum cohomology and enumerative geometry, *Comm. Math. Phys.* **164** (1994), 525-562.