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5. Differential Geometry and Global Analysis

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Labourie, François, Université Paris-Sud, France
Large Group Actions on Manifolds

In dynamical systems, one studies flows (actions of $\mathbb{R}$) or diffeomorphisms (actions of $\mathbb{Z}$). It is of course more difficult to construct an action of a general group on a manifold, with a dense orbit for instance. For finitely generated groups, the presence of many relations between the generators is obviously a strong constraint for both the existence and the deformation of an action. This elementary observation suggests that
Large group actions should be rigid (i. e. hard to deform).
An even more optimistic hope, which may work for special classes of groups (*e. g.* $SL(n,\mathbb{Z})$, for $n \geq 3$), is the following
Large group actions on a compact manifold should preserve a rigid geometric structure, at least on some open dense set.

By a geometric structure, one means a k-rigid A-geometric structure in the sense of M. Gromov. The essential feature of such a structure is that a germ of a diffeomorphism preserving it, is completely determined by its derivatives up to order k. A connection, a non degenerate metric or a projective structure definesa rigid structure, while a symplectic or a complex structure do not. To preserve a flat connection, is much the same as to be linearisable, a classical theme of dynamical systems.
To preserve a rigid geometric structure on an open dense set is a strong property as the following theorem of M. Gromov [G] shows.
Theorem *If the pseudo-group of germs of isometry has a dense orbit, then it has an open dense orbit.*
An isometry of a structure is just a diffeomorphism preserving the structure. This theorem implies that a group action with a dense orbit and preserving a geometric structure on some open dense set is, up to a technical subtelty, locally modelled on an homogeneous situation. Therefore, finding an invariant geometric structure by a group action is an important step towards a description of this action. Actually, there is no such a thing as a definition of a large group action. It is a rather vague and versatile notion. But, somehow, we have examples. Initiated by R. Zimmer's work [Z], a well studied class of large group actions is the case of volume preserving ergodic actions of Lie groups of real rank greater than 2, or lattices in these (*e. g* $SL(n,\mathbb{Z})$ for $n \geq 3$). Since then, however, other situations and groups have been studied, and we should not restrict ourselves to this class. E. Ghys and

V. Sergiescu advocate for instance the case of the Thompson group [Gh-S].

Let's now give some examples of actions of lattices, where, to avoid technical definitions, we emphasize the case of $SL(n, \mathbb{Z})$.
1- The action of $SL(n, \mathbb{Z})$ on the torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$. This action preserves a flat connection and is volume preserving and ergodic. From the dynamical point of view, an important feature is the presence of numerous Anosov diffeomorphims.
2- The action of $SL(n, \mathbb{Z})$ on the sphere S^n. This action preserves a projective structure but no measure.
3- Let G be a semisimple group containing a subgroup isomorphic to $SL(n, \mathbb{R})$. All the non compact subgroups of G, and in particular $SL(n, \mathbb{Z})$, will be acting on the right, ergodically and preserving a volume form on G/L, where L is a lattice in G. This action also preserves the Killing form of G. So far, all these actions preserve a geometric structure on the whole manifold. A new class of example have been introduced by A. Katok and J. Lewis using the following simple but useful technique [K-L].
4- The action of $SL(n, \mathbb{Z})$ on T^n has a fixed point, namely 0. We can now blow up this point, that is replace it by the projective space of its tangent space. One easily checks that $SL(n, \mathbb{Z})$ still acts on this manifold although the geometric structure is just preserved on an open dense set. Other very interesting examples have been constructed by similar techniques.

In the case of lattices, R. Zimmer's version of G. Margulis's superrigidity is a good support for both the rigidity question and the existence of an invariant geometric structure. For other groups, we have a series of positive answers for the rigidity question. In my talk, I will try to survey these theorems (due to many mathematicians) concerning analytic actions, Anosov actions, rigidity properties, and the topological structure of manifolds admitting actions of large groups.

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