ICM 1998, Berlin, Aug. 18–27

Abstracts of Plenary and Invited Lectures

Section:

5. Differential Geometry and Global Analysis

1991 MS Classification: 53 C 42
Pinkall, Ulrich, TU Berlin

Quaternionic line bundles and the differential geometry of surfaces

In this talk, which is based on joint work with Dirk Ferus and Franz Pedit, we discuss an extension of classical Riemann surface theory to what could be called "quaternionic Riemann surface theory". This extension provides a new approach to the differential geometry of surfaces immersed conformally in 3- and 4-space.

Meromorphic functions on a Riemann surface M are quotients of holomorphic sections of complex holomorphic line bundles. From the viewpoint of differential geometry they can be regarded as branched conformal immersions into the Riemann sphere. Thus, as surfaces in 3-space, they all are just the round 2-sphere. To obtain more general (branched) conformal immersions of M into 3-space we replace the complex line bundles over M by quaternionic line bundles and take quotients of "quaternionic holomorphic" sections.

The quaternionic line bundles L in this theory are all direct sums $L = E \oplus E$ where E is a complex line bundle over M. A "quaternionic holomorphic" structure on such a line bundle is given by a first order linear operator $D = d + Q$ on smooth sections of L. Here d is (the direct sum) of a $\bar{\partial}$-operator on the complex line bundle E, i. e., a holomorphic structure on E, and Q is a quaternionic linear endomorphism of L anti-commuting with the complex structure on L. The endomorphism Q is zero if and only if $L = E \oplus E$ holomorphically, in which case one simply deals with the usual theory of complex holomorphic line bundles. The length $|Q|^2$ is a 2-form and can be integrated over M to give a measure $||Q||^2$ of how far away the quaternionic holomorphic structure is from a complex holomorphic structure.

Any conformal immersion f into 3-space induces a quaternionic holomorphic line bundle so that f itself becomes a quaternionic holomorphic section. The endomorphism Q is related to the Hopf differential and $||Q||^2$ is the Willmore energy of f. Conversely, the quotients of quaternionic holomorphic sections of a quaternionic holomorphic line bundle give rise to (branched) conformal immersions into 3 and 4-space.

Enough of the basic results survive to preserve the flavour of the classical algebraic geometry of Riemann surfaces:

1. Riemann-Roch holds for quaternionic holomorphic line bundles.
2. Quaternionic holomorphic sections have isolated zeros and vanish like $z^m + O(|z|^{m+1})$. Thus we can define the vanishing order.
3. If ϕ is a non-trivial quaternionic holomorphic section of L then $\pi \deg L + ||Q||^2 \geq \pi Z$ where Z is the sum of the vanishing orders of ϕ. Equality holds if and only if the quaternionic holomorphic structure comes from a complex holomorphic structure on L^{-1}.
4. There is a correspondence between quaternionic holomorphic line bundles over M and special maps f into quaternionic projective space. These maps, called quaternionic holomorphic, are such that projections to quaternionic lines (4-spheres) are conformal.

Using the above properties we discuss several geometric results. The first is an analogue of the fact that any rational curve in complex projective space of degree 1 is a projective line: given a quaternionic holomorphic map of a 2-sphere into quaternionic projective space of degree 1 for which $||Q||^2 < 3\pi$ then the image of such a map must lie in a quaternionic line.
The second result is a rigidity result for immersed 2-spheres in 3-space: two immersed 2-spheres whose mean curvature half densities $H|df|$ are the same and whose Willmore energy is less then 16π are congruent up to scale. The bound 16π is sharp.
Finally we point out how the various "Weierstrass representations" for conformally immersed surfaces in 3-space can be obtained from special quaternionic line bundles: on the quaternionic spin bundle $L = S \oplus S$, where S is a complex spin bundle over M, we consider quaternionic holomorphic structures D for which Q is real. Then any quaternionic holomorphic section ϕ of L squares to a closed R^3-valued 1-forms ϕ^2 on M which can be integrated (up to translational periods) to a conformal immersion f of M.
More applications include results concerning Willmore surfaces, isothermic surfaces and the Bonnet problem.