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6. Topology

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Toward a global understanding of $\pi_*(S^n)$

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Determining the homotopy groups of spheres is a problem which has occupied mathematicians for over 50 years. Early work used the fibration sequences of James at the prime 2 and Toda at odd primes. These sequences have been called EHP sequences after the names of the maps which occur. At two we have

$$\cdots \to \pi_k(S^n) \to \pi_{k+1}(S^{n+1}) \to \pi_{k+1}(S^{2n+1}) \to \pi_{k-1}(S^n) \to \cdots$$

The first map the the suspension homomorphism, the second is the Hopf invariant and the third is labeled P. The key feature of this collection of ideas is that knowing $\oplus_{j<k}\pi_{j+n}(S^n)$ for all n allowed one, in principle, to calculate $\pi_{k+n}(S^n)$. Since $\pi_*(S^1)$ is known, a group by group calculation seemed possible.

¿From this approach two things emerged. The calculation was very complicated and did not seem systematic. In particular, there was not a compelling pattern connecting adjacent groups. This is in sharp contrast to the dual problem of computing, $H^{n+k}(K(\mathbf{Z},n))$, where $K(\mathbf{Z},n)$ is the Eilenberg-Mac Lane space with one non-zero homotopy group. The latter computation was possible primarily because the answer was *free*. On the other hand $\pi_*(S^n)$ is not a very nice ring in any known structure.

Recent efforts have concentrated on finding subsets or quotients of homotopy groups with some global structure. One of the early results in this direction is the theorem of Serre which determines $\pi_*(S^n) \otimes \mathbf{Q}$. The next step in this direction determined all of $\pi_*(S^n)$ which is related to K-theory. This is the work of Adams, Mahowald, Davis, Thompson and Miller and was completed nearly 20 years ago. Just as $H^{n+k}(K(\mathbf{Z},n);\mathbf{Z}/2)$ is easier to describe than these groups with integers for coefficients, so $\pi_*(S^n;\mathbf{Z}/2)$ has more structure. There is a graded ring $\mathbf{Z}[a]$ with the dimension of $a = 8$ and $\pi_*(S^n;\mathbf{Z}/2)$ is a module over this ring. The free part of this module structure is understood and this formed the key idea which allowed for the complete determination of the homotopy groups related to K-theory. In order to discuss the next case, we have to use a more complicated coefficient module. It is becoming clear what that should be. In this talk I will describe the new tools which are being used to understand the next component of homotopy and the new problems which have arisen.