ICM 1998, Berlin, Aug. 18–27

Abstracts of Plenary and Invited Lectures

Section:

6. Topology

1991 MS Classification: 57M
Ohtsuki, Tomotada, Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Oh-okayama, Meguro-ku, Tokyo 152-8552, Japan

A filtration of the set of integral homology 3-spheres

In 1989, Witten [3] proposed his famous formula of topological invariants of 3-manifolds, based on Chern-Simons gauge theory. The formula is given by using a path integral over all G connections on a 3-manifold, where G is a fixed Lie group. Following combinatorial properties of the invariants predicted by Witten's formula, the invariants, what we call the *quantum G invariant*, have been rigorously reconstructed by many researchers, say by using surgery presentations of 3-manifolds.
Since we have many Lie groups, we have obtained many quantum invariants of 3-manifolds in this decade. To control these many invariants we consider the following two approaches.

- Unify them into an invariant.
- Characterize them with a common property.

By the first approach, we expect the existence of the universal invariant among quantum invariants, though the universal quantum invariant of 3-manifolds is not found yet. Instead of universal quantum invariants, we have the universal invariant Ω among perturbative invariants of 3-manifolds, where the perturbative G invariant is obtained from the quantum G invariant by "asymptotic expansion". By the second approach, we obtain the notion of finite type such that the d-th coefficient of a perturbative invariant is of finite type of degree d. For detailed arguments see [2].

We prepare some notations. A *web diagram* on the empty set ϕ is a trivalent graph such that a cyclic order of three edges around each trivalent vertex of the graph is fixed. We denote by $\mathcal{A}(\phi;\mathbf{C})$ (resp. $\mathcal{A}(\phi;\mathbf{Z})$) the quotient vector space over $\mathbf{C}$ (resp. $\mathbf{Z}$) subject to the AS and IHX relations. We define the degree of a web diagram to be half the number of trivalent vertices of the graph. We denote by $\mathcal{A}(\phi;\mathbf{C})^{(d)}$ and $\mathcal{A}(\phi;\mathbf{Z})^{(d)}$ their subspaces of degree d. We denote by $\hat{\mathcal{A}}(\phi;\mathbf{C})$ the completion of $\mathcal{A}(\phi;\mathbf{C})$ with respect to the degree; values of the above universal invariant Ω belong to this space $\hat{\mathcal{A}}(\phi;\mathbf{C})$. We have a map $\mathcal{A}(\phi;\mathbf{C}) \to \mathbf{C}$ obtained by substituting a Lie algebra to web diagrams; we call the map *weight system.* For precise definitions of these notations see for example [2].

For simplicity we restrict our attention to integral homology 3-spheres ($\mathbf{Z}$HS's) in the following.

Theorem 1 (see [2] for precise statement)
(1)(**Ohtsuki** for $G = SO(3)$) *The invariant Ω is universal among perturbative invariants, i. e. , the perturbative G invariant recovers from the invariant Ω by the weight system derived from substitution of the Lie algebra of G.*
(2)(**Le**) *The invariant Ω is universal among all finite type invariants i. e. , the map*

$$\{\mathbf{Z}HS\text{'}s\} \xrightarrow{\Omega} \hat{\mathcal{A}}(\phi;\mathbf{C}) \xrightarrow{proj} \mathcal{A}(\phi;\mathbf{C})^{(\leq d)} \xrightarrow{W} \mathbf{C}$$

is of finite type of degree d for any linear map W, and conversely any finite type invariant of degree d is obtained as the above composed map for some linear map W.

How fine do the invariants in Theorem 1 distinguish $\mathbf{Z}$HS's? (Since it is conjectured that the perturbative G invariant dominates the quantum G invariant, this question might be equivalent to the question: how fine do quantum invariants distinguish $\mathbf{Z}$HS's?)

We define the d-th equivalence relation $\sim_d$ among $\mathbf{Z}$HS's as follows. Two $\mathbf{Z}$HS's M and M' are *d-th equivalent*, denoted by $M \sim_d M'$, if $v(M) = v(M')$ for any finite type invariant v of degree $\leq d$. We have the ascending series of the sets of the equivalence classes;

$$\{\mathbf{Z}\text{HS's}\}/\sim_1 \longleftarrow \{\mathbf{Z}\text{HS's}\}/\sim_2 \longleftarrow \{\mathbf{Z}\text{HS's}\}/\sim_3 \longleftarrow \cdots.$$

Such equivalence relations have been originally studied by Habiro by using his "claspers"; to be precise his definition of the equivalence relations is slightly stronger than ours. Habiro showed that generators of $\{\mathbf{Z}\text{HS's}\}/\sim_{d+1}$ are given by web diagrams of degree d in some sense. As a corollary of results of Habiro [1] we have

Theorem 2 (a corollary of results in [1]) *The set* $\{\mathbf{Z}\textit{HS's}\}/\sim_{d+1}$ *becomes a commutative group which is isomorphic to* $\mathcal{A}(\phi;\mathbf{Z})^{(\leq d)}_{conn}$*, where* $\mathcal{A}(\phi;\mathbf{Z})_{conn}$ *denotes the subspace of* $\mathcal{A}(\phi;\mathbf{Z})$ *spanned by connected web diagrams. By taking the direct limit of these isomorphisms we have the following homomorphism*

$$\{\mathbf{Z}\textit{HS's}\} \longrightarrow \varprojlim_{d} \{\mathbf{Z}\textit{HS's}\}/\sim_d \cong \hat{\mathcal{A}}(\phi;\mathbf{Z})_{conn}. \tag{1}$$

It is known that there exists the invariant $\omega(M) \in \mathcal{A}(\phi;\mathbf{C})_{\mathrm{conn}}$ satisfying $\Omega(M) = \exp\omega(M)$. The map (1) gives a reconstruction of the invariant ω, which dominates all invariants in Theorem 1. If we could identify $\{\mathbf{Z}\text{HS's}\}$ with a subset of $\hat{\mathcal{A}}(\phi;\mathbf{Z})_{\mathrm{conn}}$ by the map (1), all invariants in Theorem 1 would be understood as functions of weight systems on the space $\hat{\mathcal{A}}(\phi;\mathbf{Z})_{\mathrm{conn}}$.

References

[1] Habiro, K. , *Clasper Theory and its Applications*, in preparation.

[2] Ohtsuki, T. , *Combinatorial quantum method in 3-dimensional topology*, lecture note at Oiwake seminar, 1996, available at http://www.is.titech.ac.jp/labs/ohtsukilab.

[3] Witten, E. , *Quantum field theory and the Jones polynomial*, Commun. Math. Phys. **121** (1989) 360–379.