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Section:

7. Lie Groups and Lie Algebras

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Analytic structures on representation spaces of reductive groups

1. Let G be a real reductive group (for example, $G = GL(n, \mathbf{R})$). We would like to study analytic structures which are naturally defined on a representation (π, G, V) of the group G.

In order to explain what we are after let us consider first a model case. Namely, suppose we are given a C^∞ manifold M and a C^∞ vector bundle E on M and we would like to study possible analytic structures on the space V of the sections of E.
We may consider many different analytic structures on V: smooth sections , L^2 sections (or more generally, L^p sections), different kinds of Sobolev spaces of sections, of Besov spaces of sections and so on.

There is a convenient way to represent all these structures. Namely, we fix the space $V = V^\infty$ of smooth sections and study different topologies $\mathcal{T}$ on this space. For a given topology $\mathcal{T}$ we can then complete the space V with respect to $\mathcal{T}$.
For example, while the space V does not have canonical structure of a (pre)Hilbert space it clearly has a canonical structure of a Hilbertian space, i. e. a topological vector space with topology given by a norm N such that N^2 is a Hermitian form on V.
For every real number s we also can consider a topology $\mathcal{T}_s$ defined by a Sobolev norm S_s on the space V; the completion is the Sobolev space of sections V_s.

2. Now let us come back to the case of a representation (π, G, V). We assume that this representation is admissible, i. e. its restriction to the maximal compact subgroup $K \subset G$ has finite multiplicities. For simplicity of presentation let us also assume that this representation is irreducible.
By remarkable theorem of Casselman and Wallach there exists a canonical "smooth" version of this representation $(\pi, G, V = V^\infty)$. We would like to discuss some natural topologies which can be canonically constructed on V.

Consider as an example the simplest case $G = SL(2, \mathbf{R})$. In this case (for generic V) the space V can be realized as the space D_λ of smooth functions on the punctured plane $\mathbf{R}^2 \setminus 0$ homogeneous of degree $\lambda - 1$. Restricting to a unit circle $S^1 \subset \mathbf{R}^2 \setminus 0$ we get a morphism $\nu_\lambda : V \to \mathcal{F}$, where $\mathcal{F} = C^\infty(S^1)$

is the space of functions on the unit circle. Thus we can define s Sobolev structure on the space V using the norm $N_s(v) = S_s(\nu_\lambda(v))$.

The problem is that the space V has two natural realizations, as D_λ and as $D_{-\lambda}$, and the norms N_s obtained from these realizations are not equivalent. However, it is not difficult to check, that if we define the Sobolev norm N_s on the space V to be $N_s(v) = S_{s+r}(\nu_\lambda(v))$, where $r = \mathrm{Re}(\lambda)$, then the corresponding topology on the space V does not depend on the realization. This allows us to define a canonical s-Sobolev structure on the space V. In particular for $s = 0$ we get a canonical Hilbertian structure on V.

3. Let us discuss the case of a general reductive group G. For simplicity assume that G is split (e. g. $G = SL(n, \mathbf{R})$). Then again we can consider a series of representations $(\pi_\lambda, G, D_\lambda)$ parametrised by characters λ of the Cartan subgroup $A \subset G$; each of this representations can be realized in the space $\mathcal{F}$ of functions on the flag variety $X = G/B$, where B is a Borel subgroup.

We claim that the space $\mathcal{F}$ has canonical system of Sobolev norms S_s parametrized by points s of the $\mathbf{R}$ linear space $\mathfrak{a}^* = \mathrm{Mor}(A, \mathbf{R}^+)$; this space is dual to the Cartan subalgebra $\mathfrak{a} = \mathrm{Lie}(A)$.
Now, similarly to the $SL(2)$ case, we can define s Sobolev norm on the space V as $N_s(v) = S_{s+r}(\nu_\lambda(v))$, where $r = \mathrm{Re}(\lambda)$.

This construction defines a canonical system of Sobolev structures on a representation V of the group G which is parametrized by points $s \in \mathfrak{a}^*$ (at least for generic V). In particular, we define a canonical Hilbertian structure on V. In order to show that this definition extends to all admissible representations we use alternative definition of these structures based on the assymptotics of matrix coefficients.

Remarks. 1. Similar considerations suggest a conjecture that one can canonically define on V also L_p structures, L_p Sobolev structures and Besov structures.
2. If we consider representations of a p-adic reductive group G then we will find exactly the same analytic structures. They are parametrized by points s of the real vector space $\mathfrak{a}^* = \mathrm{Mor}(A, \mathbf{R}^+)$.