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Abstracts of Plenary and Invited Lectures

Section:

7. Lie Groups and Lie Algebras

1991 MS Classification:
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From double Hecke algebra to analysis

() The starting point of many mathematical and physical theories is the formula:

$$\int_{-\infty}^{\infty} e^{-x^2} x^{2k} \mathrm{dx} \;=\; \Gamma(k+1/2),\ \Re k > -1/2. \qquad (1)$$

Let us give some examples.
(a) Its direct generalization to the Bessel functions, namely, the invariance of the Gaussian e^{-x^2} with respect to the Hankel transform, is a cornerstone of the Plancherel formula.
(b) The deformation of this formula

$$\int_{-\infty}^{\infty} (e^{x^2}-1)^{-1} x^{2k} \mathrm{dx} \;=\; \Gamma(k+1/2)\zeta(k+1/2),\ \Re k > 1/2, \qquad (2)$$

is fundamental in the analytic number theory.
(c) Replacing x^{2k} by $\sinh(x)^{2k}$, we come to the theory of spherical and hypergeometric functions and to the spherical Fourier transform. The spherical transform of the Gaussian plays an important role in the harmonic analysis on symmetric spaces.
To employ modern mathematics at full potential, we do need to go from Bessel to hypergeometric functions. In contrast to the former, the latter can be studied, interpreted and generalized by a variety of methods in the range from representation theory and algebraic geometry to integrable models and string theory. However straightforward substituting $x^{2k} \to \sinh(x)^{2k}$ creates problems. The formula (1) has no sinh-counterpart, the Gaussian looses its Fourier-invariance, and the spherical transform is not selfdual anymore.
Difference setup. It was demonstrated recently that we can save these and related properties switching to the truncated theta-function:

$$\mu_k(x;q) \;=\; \prod_{j=0}^{\infty} \frac{(1-q^{j+2x})(1-q^{j+1-2x})}{(1-q^{j+k+2x})(1-q^{j+k+1-2x})},\ 0<q<1,\ k\in \mathbf{C}. \qquad (3)$$

Actually it is a unification of $\sinh(x)^{2k}$ and the Harish-Chandra function (A_1) serving the inverse spherical transform. So it is not surprising that the μ-transform is selfdual and has other remarkable properties. For instance, setting $q = \exp(-1/a),\ a > 0,$

$$(-i)\int_{1/4-\infty i}^{1/4+\infty i} q^{-x^2}\mu_k \mathrm{dx} = \sqrt{a\pi}\prod_{j=1}^{\infty}\frac{1-q^{j+k}}{1-q^{j+2k}} \tag{4}$$

provided that $\Re k > -1/2$. The limit of (4) multiplied by $(a/4)^{s-1}$ as $a \to \infty$ is precisely the imaginary variant of (1).
Once we managed Γ, it would be unexcusable not to try

$$\widetilde{\zeta}_q(s) = \ (-i)\int_{1/4-\infty i}^{1/4+\infty i} (q^{x^2}-1)^{-1}\mu_k \mathrm{dx} \text{ for } k = s-1/2,\ \Re s > 0. \tag{5}$$

The limit of $(a/4)^{s-1}\widetilde{\zeta}_q(s)$ as $a \to \infty$ is $\Gamma(s)\zeta(s)$ for $\Re s > 1$. In the critical strip, it is different:

$$\lim_{a\to\infty} a^{2s-2}\widetilde{\zeta}_q(s) \ = \ -\cot(\pi s)\Gamma^2(s-1/2) \ \text{ for } 0 < \Re s < 1. \tag{6}$$

However the zeros of ζ have q-deformations with interesting properties.
Double Hecke algebras provide justifications and generalizations. In the A_1-case, $\mathcal{H}$ is the quotient of the group algebra of

$$\Pi_q = \langle T, X, Y, q^{1/4}\rangle,\ TXTX = 1 = T^{-1}YT^{-1}Y,\ Y^{-1}X^{-1}YXT^2 = q^{1/2}, \tag{7}$$

by $(T-t^{1/2})(T+t^{-1/2}) = 0$. So the definition involves central q, t (and their roots). Renormalizing $T \to q^{1/4}T,\ X \to q^{-1/4}X,\ Y \to q^{1/4}Y^{-1}$,

$$\Pi_q \cong \Pi_1 \mod \langle q^{1/4}\rangle,\ \Pi_1 \cong \pi_1(\{E\times E\setminus \mathrm{diag}\}/\mathbf{S}_2),\ E = \text{ elliptic curve}, \tag{8}$$

a special case of the calculation due to Birman.
Thanks to the topological interpretation, $PSL_2(\mathbf{Z})$ acts projectively on Π_1 and $\mathcal{H}$. For instance, the automorphism τ sending $Y \to q^{1/4}XY$ and fixing X, T, q, t corresponds to the standard parabolic generator.
The automorphism τ becomes the conjugation by the q-Gaussian q^{-x^2} in the following $\mathcal{H}$-representation on functions of x :

$$T \to t^{1/2}s + (q^{2x}-1)^{-1}(t^{1/2}-t^{-1/2})(s-1),\ X \to q^x,\ Y \to spT \tag{9}$$

for the reflection $sf(x) = f(-x)$ and the translation $pf(x) = f(x+1/2)$. The Fourier transform is associated with the anti-involution $\{\sigma : X \to Y^{-1} \to X\}$ preserving T, t, q.

Diagonalizing Y and its multi-dimensional counterparts we come to the Macdonald polynomials. Combining τ and σ, we prove the Fourier-invariance of these polynomials multiplied by the Gaussian and get (4). Switching to the roots of unity, we generalize the Verlinde algebras and the Gauss sums. Last but not the least, we obtain the classical p-adic theory of spherical functions (Iwahori, Matsumoto, Macdonald) as $q \to 0$.