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Section:

8. Analysis

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The Baum-Connes Conjecture

Let G be a second countable, locally compact group and denote by $C^*_{\mathrm{red}}(G)$ the reduced C^*-algebra of G. It is the completion in the operator norm of the convolution algebra $L^1(G)$, acting as operators on $L^2(G)$. The Baum-Connes conjecture proposes a means of calculating the C^*-algebra K-theory groups $K_*(C^*_{\mathrm{red}}(G))$ using group homology and the representation theory of compact subgroups of G. It originates in work of Kasparov and Mishchenko on the Novikov higher signature conjecture and ideas of Connes in foliation theory. The first cases of the conjecture were settled by Pimsner and Voiculescu, who investigated free and free abelian groups, and by Connes and Kasparov, who independently considered simply connected solvable groups. These results actually preceded the current formulation of the conjecture. The conjecture is true, and essentially trivial, for compact groups.
The formulation of the Baum-Connes conjecture requires Kasparov's equivariant KK-theory, using which equivariant K-homology theory for G-spaces may be developed. If $\underline{E}G$ denotes the universal proper G-space then the conjecture asserts that a certain natural assembly map

$$\mu\colon K^G_*(\underline{E}G) \to K_*(C^*_{\mathrm{red}}(G))$$

is an isomorphism. If G is a discrete and torsion-free group (this is the most interesting and important case) then $K^G_*(\underline{E}G)$ is the K-homology of the classifying space BG. In this situation the resulting assembly map

$$\mu\colon K_*(BG) \to K_*(C^*_{\mathrm{red}}(G))$$

is very closely related to its namesake in manifold theory. Indeed the Baum-Connes conjecture is a close cousin of the manifold-theoretic Borel conjecture and it actually implies the Novikov conjecture. These and other relations to topology are the cause a good deal of interest in the Baum-Connes conjecture among topologists and homotopy theorists.
In recent joint work with Kasparov the author has proved the following result. A *metrically proper* action of a locally compact group on Hilbert space is a continuous action by affine isometries such that $\|g \cdot v\| \to \infty$ as $g \to \infty$, for all vectors v.

Theorem The Baum-Connes conjecture holds for all groups which admit metrically proper actions on Hilbert space.

The theorem applies to amenable groups and also unifies a good deal of what is currently known about the Baum-Connes conjecture. On the other hand it applies to no non-compact group with Kazhdan's property T. Indeed the conjecture is currently known for no such discrete group, although by invoking a great deal of tempered representation theory the conjecture has been proved by Wassermann for linear reductive Lie groups. Future progress for discrete groups will probably require a penetrating analysis of the relationship between the harmonic analysis of discrete groups and K-theory for group C^*-algebras.