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Section:

8. Analysis

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Rectifiability, analytic capacity, and singular integrals

This lecture is a survey on some interplay between geometric measure theory (rectifiability), complex analysis (analytic capacity) and harmonic analysis (singular integrals). The theory we shall discuss has been mainly developed during this decade. Vaguely, it deals with the following three principles:

1. Singular integrals of degree m behave nicely on an m-dimensional subset E of $\mathbb{R}^n$ if and only if E is in some sense rectifiable.
2. The analytic capacity of a 1-dimensional compact subset E of the complex plane is zero if and only if E is purely unrectifiable.
3. The analytic capacity of a 1-dimensional compact subset E of the complex plane is positive if and only if the Cauchy singular integral operator is L^2-bounded in a large part of E.

By the m-dimensionality of E we usually mean that the m-dimensional Hausdorff measure $H^m(E)$ is positive and finite. The rectifiability of a set means that H^m almost all of it can be covered with m-dimensional C^1-surfaces. The nice behaviour of singular integrals in 1 could refer to L^2-boundedness or the almost everywhere existence of principal values. In both cases there are theorems with some extra assumptions on the set E in question. A typical here relevant singular integral operator T is that related to the Riesz kernel:

$$Tg(x) = \int_E |x-y|^{-m-1}(x-y)g(y)dH^m y, x \in \mathbb{R}$$

A compact subset of the complex plane $\mathbb{C}$ has analytic capacity zero if and only if it is removable for bounded analytic functions, almost by definition. The highlight of this topic is that the principle 2 holds very generally. That is, we have the following recent result of G. David:

Theorem. *Let E be a compact subset of $\mathbb{C}$ with $H^1(E) < \infty$. Then the analytic capacity of E is zero if and only if E is purely unrectifiable, that is, $H^1(E \cap \Gamma) = 0$ for every rectifiable curve Γ.*

The proof of this result combines principles 1 and 3: if E has positive analytic capacity, the Cauchy operator is bounded on a large part of E, by a very general $T(b)$-theorem which David proves. By principle 1 this large part is rectifiable, and so E is not purely unrectifiable.

The main tool used in the proof is the following remarkable identity relating Menger curvature to the Cauchy kernel. It was found by M. S. Melnikov for a few years ago. The Menger curvature $c(z_1, z_2, z_3)$ of the triple $z_1, z_2, z_3 \in \mathbb{C}$ is the reciprocal of the radius of the circle passing through these points. The identity is

$$c(z_1, z_2, z_3)^2 = \sum_\sigma \frac{1}{(z_{\sigma(1)} - z_{\sigma(3)})\overline{(z_{\sigma(2)} - z_{\sigma(3)})}}$$

where σ runs through all six permutations of $\{1, 2, 3\}$. Melnikov and Verdera used this identity to give a simple geometric proof of the L^2-boundedness of the Cauchy singular integral on Lipschitz graphs.

There is a very similar removability problem as 2 which concerns Lipschitz harmonic functions. In the plane the exact analogue of David's theorem holds. The results in higher dimensions on that question as well as on principle 1 are much weaker, since we don't have anything like the above identity available.

Much of the standard Calderón-Zygmund-theory of singular integrals holds for general measures which satisfy the doubling condition, as has been known for a long time. Interesting recent work of F. Nazarov, S. Treil and A. Volberg, jointly, and of X. Tolsa shows that very large part of this theory is valid without any doubling condition at all. This work has been inspired by and related to the above three principles. In fact, the method developed by Nazarov, Treil and Volberg leads to a simpler proof of David's theorem.

Here are a few of the many relevant references.

G. David, *The unrectifiable 1-sets have vanishing analytic capacity*, to appear in Rev. Mat. Iberoamericana.

P. Mattila, M. S. Melnikov and J. Verdera, *The Cauchy integral, analytic capacity, and uniform rectifiability*, Ann. of Math. 144(1996), 127–136.

F. Nazarov, S. Treil and A. Volberg, *Cauchy integral and Calderón-Zygmund operators on nonhomogeneous spaces*, Internat. Math. Res. Notices 15(1997), 703–726.

X. Tolsa, *L^2-boundedness of the Cauchy integral operator for continuous measures*, to appear in Duke Math. J.