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Abstracts of Plenary and Invited Lectures

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8. Analysis

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L^p-functional calculus on Lie groups and estimates for wave equations

Consider a "sum of squares" operator

$$L = -\sum_{j=1}^{k} X_j^2$$

on a manifold M of dimension $d \geq k$, where $X_1, \ldots, X_k$ are smooth, real vector fields on M satisfying the following bracket condition: $X_1, \ldots, X_k$, together with their iterated commutators $[X_{j_1}, [X_{j_2}, [\ldots [X_{j_{\ell-1}}, X_{j_\ell}] \ldots]]]$, span the tangent space of M at every point of M. If $k = d$, then L might for example be a Laplace-Beltrami operator. If $k < d$, then L will not be elliptic, but, according to a celebrated theorem of L. Hörmander, it will still be hypoelliptic. Operators of this type arize in various contexts, for instance in higher dimensional complex analysis. Assume in addition that L is essentially selfadjoint on $C_0^\infty(M)$ with respect to some volume element dx. Then the closure of L, again denoted by L, admits a spectral resolution $L = \int_0^\infty \lambda dE_\lambda$ on $L^2(M)$, and any function $m \in L^\infty(\mathbb{R}^+)$ leads to an L^2-bounded operator

$$m(L) = \int_0^\infty m(\lambda) dE_\lambda.$$

An important question is then under which additional conditions on the spectral multiplier m the operator $m(L)$ extends from $L^2 \cap L^p(M)$ to an L^p-bounded operator, for a given $p \neq 2$. If so, m is called an L^p-multiplier for L.

Since, without additional properties of M and L, there is little hope in finding answers to this questions, we shall assume that M is a Lie group G, with right-invariant Haar measure dx, and that $X_1, \ldots, X_k$ are left-invariant vector fields which generate the Lie algebra of G.

A main objective of the talk will be to survey some of the relevant developments concerning this question, and moreover to link it to questions concerning estimates for the associated *wave equation*, more precisely the following Cauchy-problem:

$$\begin{aligned} \left(\tfrac{\partial^2}{\partial t^2} + L\right) u(x,t) &= 0 \quad \text{on} \quad G \times \mathbb{R}, \\ u(x,0) = f(x), \ \tfrac{\partial u}{\partial t}(x,0) &= 0, \end{aligned} \tag{1}$$

whose solution is given by $u(\cdot,t) = \cos(t\sqrt{L})f$.
The classical model case is the Laplacian $L = -\Delta$ on $\mathbb{R}^d$. Here, $m(L)$ is a Fourier multiplier operator $\widehat{m(L)f}(\xi) = m(|\xi|^2)\hat{f}(\xi)$, and a sufficient condition for m to be an L^p-multiplier for $-\Delta$ follows from a well-known multiplier theorem going back to J. Marcinkiewicz, S. Mikhlin and L. Hörmander. This "MMH-theorem" states that $m(-\Delta)$ is L^p-bounded for all $1 < p < \infty$ (and of weak type (1,1)), provided that m is locally differentiable of degree α (in the sense that $m \in H^{\alpha}_{loc}(\mathbb{R}^+)$) for some $\alpha > d/2$, in a certain sense "uniformly on every scale". This result is sharp with respect to the critical degree $d/2$ only for $p = 1$; for $1 < p < \infty$, less restrictive conditions follow by suitable interpolation with the trivial L^2-estimate.
Theorems of "MMH-type" have meanwhile been proved by various authors for groups G of polynomial volume growth. For instance, for stratified nilpotent groups, M. Christ has proved such a theorem under the condition $\alpha > Q/2$, where Q is the so-called homogeneous dimension of the group G. If G is non-abelian, then $Q > d$, where d is the dimension of G as a manifold. These results make use of estimates for solutions of the *heat equation* $(\frac{\partial}{\partial t} + L)u = 0$ associated to L, sometimes for complex time parameters t. More recently, it was proved by E. M. Stein and myself, and independently by W. Hebisch, that there exist non-abelian groups, the so-called Heisenberg groups, on which the MMH-theorem does in fact hold under the weaker condition $\alpha > d/2$. It is not known whether this condition is sufficient on all groups of polynomial growth.
In another recent joint article, the following estimate for solutions to (1) has been established for the so-called sub-Laplacian L on a Heisenberg group:

$$||u(\cdot,t)||_{L^1} \leq C(1+|t|)^{\alpha}||(1+L)^{\alpha/2}f||_{L^1}, \tag{2}$$

for any $\alpha > (d-1)/2$.
This is the analogue of the corresponding estimate by A. Miyachi and J. Peral for the classical wave equation on Euclidian space. By some subordination principle it implies the sharp multiplier result for Heisenberg groups. Extensions of (2) to more general classes of groups would therefore be desirable.
In the last part of the talk, I shall concentrate on the Laplacian $L = -\Delta$ on $\mathbb{R}^d$. Then $m(L)$ corresponds to the radial Fourier multiplier $\xi \mapsto m(|\xi|^2)$. For such radial multipliers and $p \neq 2$, better L^p-estimates can be proved than those obtained from interpolating the MMH-estimate with the trivial L^2-estimate, by making use of the curvature of the sphere $|\xi| = 1$. The famous *Bochner-Riesz-conjecture*, which is concerned with Fourier multipliers

of the form $m_\beta(\xi) = (1-|\xi|^2)_+^\beta$, describes the expected minimal smoothness conditions for these multipliers. This conjecture is again linked to the wave equation via the stronger *local smoothing conjecture*, due to C. Sogge, according to which the solution to (1) satisfies

$$||u||_{L^{p_d}(\mathbb{R}^d\times[1,2])} \leq C||(1-\Delta)^\varepsilon f||_{L^{p_d}(\mathbb{R}^d)}, \tag{3}$$

for all $\varepsilon > 0$, with $p_d = 2d/(d-1)$. The analogue of (3), in which the full L^p-norm on $\mathbb{R}^d$ is replaced by a mixed $L^p(L^2)$-norm in polar coordinates, has recently been proved in joint work with A. Seeger.
It is known that the validity of the local smoothing conjecture would imply that of other outstanding conjectures in Fourier analysis, like the "restriction conjecture" or the "Kakeya conjecture".