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Abstracts of Plenary and Invited Lectures

Section:

8. Analysis

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Discrete Analogues of Singular and Maximal Radon Transforms

For each x in R^ℓ, let $\gamma(x,t)$ be a smooth k-dimensional surface through x in R^k. That is $\gamma(x,t)$ is a smooth mapping from $R^\ell \times R^k$ into R^ℓ with $\gamma(x,0)=x$. Let $f(x)$ be a function on R^ℓ. Then for suitably small η we define the singular and maximal Radon transforms respectively as

$$Sf(x) = \int_{|t|\le\eta} f(\gamma(x,t))K(t)dt$$

and

$$Mf(x) = \sup_{0<\epsilon\le\eta} \frac{1}{|B(\epsilon)|}\int_{|t|\le\epsilon} |f(\gamma(x,t))|dt$$

where K is a Calderón-Zymund kernel on R^k, and $|B(\epsilon)|$ denotes the measure of the ball of radius ϵ in R^k. The following is a rough statement of the result we have in mind for singular and maximal Radon transforms.

Theorem 1: (Christ, Nagel, Stein and Wainger) S and M are locally bounded in L^p, $1<p<\infty$, provided $\gamma(x,t)$ satisfies an appropriate curvature hypothesis.

Corollary: If $\gamma(x,t)$ is real analytic

$$\lim_{\epsilon\to 0}\frac{1}{|B(\epsilon)|}\int_{|t|<\epsilon} f(\gamma(x,t))dt \to f(x) \text{ a. e.}$$

for f in $L^p(R^\ell)$.
We are interested in discrete analogues of Theorem 1: We let P be a polynomial mapping from $R^\ell\times R^k \to R^\ell$ with integer coefficients. Denote by Z^ℓ and Z^k the lattice points in R^ℓ and R^k, that is points with integer coordinates. For $m\in Z^\ell$, set

$$\mathcal{S}f(m) = \sum_n{}' f(P(m,n))K(n)$$

where the prime denotes the term $n = 0$ is to be omitted, and

$$\mathcal{M}f(m) = \sup_N \frac{1}{N^k} \sum_{|n| \leq N} |f(P(m,n))|.$$

We are interested in estimates for $\mathcal{S}$ and $\mathcal{M}$ on $\ell^p(\mathbb{Z}^\ell)$. There are the following results:

Theorem 2: (Arkhipov and Oskolkov) (1987) If

$$P(m,n) = m + Q(n)$$

$$\|\mathcal{S}f\|_{\ell^2} \leq A\|f\|_{\ell^2}.$$

Theorem 3: (Bourgain) (1988-1989) If

$$P(m,n) = m + Q(n)$$

$$\|\mathcal{M}f\|_{\ell^p} \leq A\|f\|_{\ell^p}, \quad 1 < p \leq \infty.$$

Theorem 4: (Stein and Wainger) (1990) If

$$P(m,n) = m + Q(n)$$

$$\|\mathcal{S}f\|_{\ell^p} \leq A\|f\|_{\ell^p}, \frac{3}{2} < p < 3.$$

Theorem 5: (Stein and Wainger) (1997) Let

$$m = (m', m'') \quad \text{with} \quad m' \in z^k, m'' \in z^{\ell-k}.$$

If

$$P(m,n) = (m' + n, m'' + Q(m', n))$$

$$\|\mathcal{S}f\|_{\ell^2} \leq A\|f\|_{\ell^2}.$$

A related theorem is due to M. Wierdl.

Theorem 6: (1988) Let p_n denote the nth prime, and for m in Z set

$$Mf(m) = \sup_R \frac{1}{R} \sum_{n=1}^{R} |f(m - p_n)|.$$

Then

$$\|Mf\|_{\ell^p} \leq A\|f\|_{\ell^p}, 1 < p \leq \infty,$$

Theorems 3) and 6) have applications to ergodic theory.
In our lecture we shall discuss the difficulties that arise in attacking the discrete problems and how methods of analytic number theory are used in the attack.