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Section:

9. Ordinary Differential Equations and Dynamical Systems

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Reducibility and point spectrum for linear quasi-periodic skew-products

A linear quasi-periodic skew-product is a dynamical system

$$(X, \theta) \to (A(\theta)X, \theta + \omega),$$

where $A(\theta)$ is a matrix valued function on the torus $\mathbb{T}^d$ and ω is a vector in $\mathbb{R}^d$. $A(\theta)$ may or may not be given as a "time-1-map" of a system of differential equations.
If $\{n\omega\}_{n\in\mathbb{Z}}$ is discrete in $\mathbb{T}^d$, the periodic case, then this system behaves basically as a constant system — it is *reducible* in some sense. In particular, there are no solutions decaying both in forward and backward time. If we think of the dynamical system as an operator on doubly infinite sequences in l^2 this means that 0 is not a *point eigenvalue* and, hence, there are no *point spectrum* at all.
If $\{n\omega\}_{n\in\mathbb{Z}}$ is dense in $\mathbb{T}^d$ the situation is very different and far from being completely understood. Best understood is when ω is Diophantine, i. e.

$$| < k, \omega > | \ \geq \ \frac{\kappa}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^d \setminus 0,$$

for some constants $\kappa > 0$ and $\tau > d-1$. Then the system can be studied by perturbation theory of KAM-type and it has been known since the 60's that "most" systems sufficiently close to constants are reducible. These results did not however apply to all, not even to "almost all", systems close to constants.
In the late 80's a perturbation theory was developed for the other extreme. Fröhlich-Spencer-Wittver and Sinai, independently, were able to prove that certain Schrödinger equations

$$-(u_{n+1} + u_{n-1}) + V(\theta + n\cdot\omega)u_n \ = \ Eu_n$$

sufficiently far from constants had pure point spectrum. This implies that the corresponding skew-product

$$A(\theta) \ = \ \begin{pmatrix} 0 & -1 \\ 1 & V(\theta) - E \end{pmatrix}$$

have point eigenvalues for all values of E in the spectrum.
In recent years these methods have been improved and, in particular, $SL(2,\mathbb{R})$ — related to the the Schrödinger equation — and $SO(3,\mathbb{R})$ have been well studied. We now have a fairly complete picture of the "close-to-constant" situation: the reducible systems are typical in measure theoretical sense — "almost all" systems are reducible; non-reducible systems are typical in a topological sense — generic systems are non-reducible. In $SO(3,\mathbb{R})$ the generic system is even uniquely ergodic as a dynamical system on $SO(3,\mathbb{R}) \times \mathbb{T}^d$.
In the "far-from-constant" situation the picture is less complete but also here there has been some progress: pure point spectrum is typical for a very large class of Schrödinger equations when $d = 1$.

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