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Section:

9. Ordinary Differential Equations and Dynamical Systems

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Random Dynamics and its Applications

It is not always realistic to describe evolution of a physical system by repeated applications of exactly the same transformation. Thus the need to study actions of compositions of different maps (time dependent transformations) emerges in a natural way. The framework of random transformations is quite appropriate for this purpose if we have in mind results which hold true for a typical in some sense sequence of maps. Stochastic flows appearing as solutions of stochastic differential equations give another important motivation to deal with random transformations.
The set up consists of a probability space $(\Omega, \mathcal{A}, P)$ together with a $P-$preserving invertible map $\theta : \Omega \to \Omega$ and of a measurably depending on ω family of (in general, different) spaces X^ω together with a family of transformations $F_\omega : X^\omega \to X^{\theta\omega}$. The object of study is the action of compositions $F^n_\omega = F_{\theta^{n-1}\omega} \circ \cdots \circ F_{\theta\omega} \circ F_\omega$. The main notions of deterministic dynamics such as invariant measures and invariant sets are being replaced here by random "invariant" measures μ^ω on X^ω and random "invariant" sets $U^\omega \subset X^\omega$ satisfying $F_\omega \mu^\omega = \mu^{\theta\omega}$ and $F_\omega U^\omega = U^{\theta\omega}$. A suitable analogy of the Kolmogorov-Sinai entropy, called fiber or relativized entropy, appears in this set up, as well. In the case of smooth random transformations further developments lead to random "invariant" subbundles and submanifolds, the stochastic bifurcation theory, Lyapunov exponents and their relations with the fiber entropy, the dimension theory of random "invariant" sets and measures etc.
Among most interesting questions emerging in this framework is whether for a "typical" ω the compositions $F^n_\omega, n = 1, 2, ...$ act in a chaotic (stochastic) fashion. A similar question can be asked for compositions $F^n = F_{n-1} \circ \cdots \circ F_1$ from a given sequence $F_1, F_2, ...$ of deterministic transformations. Familiar signs of stochastic behavior are the central limit theorem, the law of iterated logarithm, large deviations type results etc. Namely, given a random measure μ^ω and a random function $g_\omega = g_\omega(x)$ with $\int g_\omega d\mu^\omega = 0$ consider $S_\omega(n, x) = \sum_{k=0}^{n-1} g_{\theta^k\omega}(F^k_\omega x)$ and veryfy whether for a typical ω the $\mu^\omega-$distribution (in $x \in X^\omega$) of $n^{-1/2} S_\omega(n, x)$ is close to normal, $S_\omega(n, x)$ has the order of $(2n \log\log n)^{1/2}$ for $\mu^\omega-$almost all x and the $\mu^\omega-$measure of points x for which $|n^{-1} S_\omega(n, x)| > \delta > 0$ is exponentially small in n.

This sort of stochastic behavior can be obtained for some classes of random transformations such as random expanding maps, random subshifts of finite type and certain random hyperbolic diffeomorphisms. For these transformations a version of random thermodynamic formalism can be developped yielding a rich collection of random "invariant" Gibbs measures μ^ω, in particular, a random analogy of the Sinai-Ruelle-Bowen measure, which describe the stochastic behavior indicated above.

The theory has nice applications to randomly evolving graphs $G(\omega)$ which are determined by a random incidence matrix $\Pi(\omega) = (\pi_{ij}(\omega), i = 1, ..., m(\omega); j = 1, ..., m(\theta\omega))$ so that $i_0, i_1, ..., i_n$ is a path in the graph $G(\omega)$ if $\pi_{i_k i_{k+1}}(\theta^k\omega) = 1$ for all $k = 0, ..., n-1$. Then the spaces X^ω above coincide with the sets of all infinite (one or two sided) paths in $G(\omega)$ with F_ω acting as the left shift. Employing the thermodynamic formalism for random subshifts of finite type it is possible to study statistical characteristics of paths in $G(\omega)$ with specific properties.

Random versions of Perron-Frobenius theorem for random positive matrices and Markov chains in random dynamical environments are also closely related to this topic. The latter is connected also with random walks on groups with stationary changing distributions which leads to random "harmonic" functions and random boundaries. Existence of nonconstant bounded random "harmonic" functions depends on positivity of certain fiber entropy type quantity which emerges also in computations of Hausdorff dimensions of random "harmonic" measures.

Another application of this theory yield computations of fractal dimensions of random sets U^ω. Consider, for instance, a random positive integer valued function $m = m(\omega)$ and the corresponding "random base" expansion of numbers $x \in [0,1]$ given by $x = \sum_{k=0}^{\infty}(m(\omega)m(\theta\omega)\cdots m(\theta^k\omega))^{-1}x_k$, $x_k \in \{0, 1, ..., m(\theta^k\omega)-1\}$. Specifying digits which can appear in such expansions or prescribing frequencies of digits we obtain random Cantor and other sets U^ω which are "invariant" under F_ω given by $F_\omega x = m(\omega)x \pmod 1$. The Hausdorff and other dimensions of such random sets can be obtained via random ergodic theory characteristics of F_ω.

Other applications of the above set up include some statistical mechanics models such as spin glasses type systems etc.