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9. Ordinary Differential Equations and Dynamical Systems

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Kuksin, Sergei B, Steklov Institute, Moscow, Russia, and Heriot-Watt University, Edinburgh, UK

Towards a Qualitative Theory of Hamiltonian PDEs

In our lecture we review some recent results on qualitative theory of Hamiltonian PDEs and related equations, assuming that the space-variable x belongs to a bounded domain (so-called "finite volume case"). We treat the equations as dynamical systems in a functions space formed by functions of x.

We start with equations which are Hamiltonian perturbations of an integrable PDE (such as the Korteweg-de Vries equation). Due to the finite-volume assumption we have imposed, these PDEs can be written as infinite chains of Hamiltonian equations. The most famous result on qualitative behaviour of a *finite* Hamiltonian chain, close to integrable, is the KAM-theorem. The theorem deals with solutions of a perturbed system which behave in the same way as solutions of the unperturbed integrable one and proves that these solutions occupy most part of the phase space when the perturbation tends to zero. This theorem admits a non-trivial and non-straightforward version, a KAM theorem for PDEs, which states that: *solutions of a perturbed integrable PDE, behaving in an integrable fashion, are asymptotically dense in a function space when the perturbation tends to zero.*

Closures in a function phase-space of solutions, constructed by the KAM theorem for PDEs, form finite-dimensional tori, invariant for the perturbed equation. They manifest rigidity of this equation's flow. In the finite-dimensional case *all* Hamiltonian equations possess some nontrivial rigidity, first observed by M. Gromov: the flow-maps of the equations preserve a non-trivial characteristic of subsets of the phase space, known as the symplectic capacity. In particular, the flow-maps cannot send a big ball into a narrow cylinder since symplectic capacity of the former is bigger than of the latter; this is Gromov's non-squeezing property. To discuss whether this property remains true for Hamiltonian PDEs, we first have to understand which norm among infinitely many non-equivalent norms of the space of functions should be used to measure the ball's radius. The right norm, a "Darboux norm of the function space", turns out to be a norm of low

smoothness (in all examples it is weaker than the C^1-norm). Hence, *Gromov's property holds true for the flow, formed by generalised solutions of a Hamiltonian PDE, provided that the flow is well defined.* A flow formed by *classical* solutions acts in a space, formed by sufficiently smooth functions. For this flow Gromov's property is wrong. This happens due to an important phenomenon, discussed below.
Let us consider the following class of PDEs:

$$\begin{aligned}&\langle\text{non-linear homogeneous Hamiltonian equation}\rangle\\ &\qquad + \langle\delta_1\text{-small linear damping}\rangle\\ &\qquad + \langle\delta_2\text{-small linear dispersion}\rangle = \langle\text{order-one forcing}\rangle. \qquad (1)\end{aligned}$$

If $\delta_1 = 0$, then this equation is Hamiltonian. Still, the most important are equations (1) with small non-zero δ_1 since exactly equations of this form describe turbulence in different physical media. Let us denote $\delta = (\delta_1^2 + \delta_2^2)^{1/2}$. The following property holds for a large class of equations (1): *solutions of* (1) *develop a short space-scale of size* $\delta^\gamma, \gamma > 0$. The exponent γ may be estimated from below and from above. This property admits an elegant reformulation in spectral terms, related to some classical heuristic results from the theory of turbulence. Proof of this short-scale behaviour of solutions is based on a "non-Gromov" property of the flow, defined by equation (1) with zero r. h. s. in the space of smooth functions: *in finite time, the flow pulls the whole phase-space through its very narrow subset, formed by fast oscillating functions.*
The three groups of results sketched above are surrounded by an ocean of open problems. Some of them will be discussed in the lecture.