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Counterexamples to the Seifert Conjecture

In 1950, H. Seifert proved the existence of a closed orbit for a flow which in some sense approximates the Hopf fibration on the 3-dimensional sphere. This result gave rise to the question as to whether every vector field on S^3 must have a periodic orbit. The assertion that such an orbit always exists became later known as the Seifert conjecture and was believed to be true until a surprising counterexample was given by P. A. Schweitzer. The idea was to locally perturb a non-singular vector field having a discrete collection of circular orbits to eliminate these circles without forming new ones. Schweitzer achieved that by constructing an aperiodic C^1 plug that allows control over the global configuration of orbits.
A sequence of fundamental results came about as a consequence of the work of A. Weinstein, P. Rabinowitz, C. Viterbo, H. Hofer, E. Zehnder, and others. Of particular importance is the 1987 paper by C. Viterbo with a proof of the Weinstein conjecture in $\mathbb{R}^{2n}$: a hypersurface of contact type carries a closed characteristic. In relation to both the Seifert and the Weinstein conjectures, a momentous advancement was made in 1993 by H. Hofer who proved the existence of a closed orbit for a smooth Reeb vector field on S^3. Subsequently, extensions of Hofer's result for S^3 (for instance, concerning unknotted orbits) were contributed by H. Hofer, K. Wysocki, and E. Zehnder. J. Etnyre and R. Ghrist proved a version of the Seifert conjecture in hydrodynamics.
In this abstract, attention is paid mainly to counterexamples to the Seifert conjecture, i. e., constructions of non-singular vector fields without circular orbits on S^3, or, almost equivalently, on any closed 3-manifold, and on some higher-dimensional manifolds. All of these examples are based on a construction of an appropriate plug–a flow exhibiting a special kind of symmetry, defined on a chart that is inserted in a manifold. This construction was first used by F. W. Wilson in the mid 1960's, and then by P. A. Schweitzer in 1972, who for the first time used an aperiodic plug to break circular orbits.
There are two types of examples of aperiodic flows on S^3 directly strengthening Schweitzer's construction. One, a $C^{3-\epsilon}$ flow, was given by J. Harrison in the mid 1980's. The other, a volume preserving C^1 plug, was constructed

by G. Kuperberg in 1995, who also proved that every 3-manifold possesses a C^∞ volume preserving flow with no fixed points and with a discrete collection of closed orbits. Thus, by inserting the plug, he shows that every 3-manifold admits a C^1 volume preserving aperiodic flow.

The examples given by Schweitzer, Harrison, and G. Kuperberg have only non-compact orbits, but they have at least two minimal sets of covering dimension 1. Earlier F. W. Wilson showed a way of modifying smooth non-singular vector fields so that there are only minimal sets of codimension 2. This gives the existence of aperiodic flows on S^{2n+1}, $n \geq 2$. A modified Seifert conjecture asserted that on certain manifolds flows must have minimal sets of codimension more than one, in particular, that every vector field on S^3 has either singular points or 1-dimensional minimal sets.

A C^∞ (actually real analytic, as noted by W. Thurston) counterexample to the Seifert conjecture was given by K. Kuperberg in 1993. Extending the construction, G. Kuperberg and K. Kuperberg solved the modified Seifert conjecture showing that S^3 admits a C^ω flow with one minimal set only, of dimension 2. In general, a foliation of any codimension of any manifold can be modified in an analytic fashion so that all minimal sets have codimension 1. It is not known whether S^3 admits a C^3 aperiodic vector field with 1-dimensional minimal sets only. It is worth mentioning that these constructions do not provide volume-preserving flows even if the differentiability requirement is relaxed to C^1.

An interesting category is that of piece-wise linear flows. G. Kuperberg and K. Kuperberg proved for any $1 \leq k \leq n-1$, every 1-foliation of an n-manifold, $n \geq 3$, can be modified in a piece-wise linear fashion so there are no closed leaves but all minimal sets are k-dimensional. G. Kuperberg proved that every 3-manifold possesses a transversely measured 1-dimensional piece-wise linear foliation with discrete closed leaves.

The Hamiltonian version of the Seifert conjecture in dimension 3 has not been solved, but there are interesting examples in higher dimensions. In 1994, V. Ginzburg and M. Herman independently constructed examples of smooth compact hypersurfaces without closed characteristics in $\mathbb{R}^{2n}$, $n \geq 4$, resolving the case of Hamiltonian flows on spheres of dimension 7 or higher. At the same time, M. Herman found a $C^{3-\epsilon}$ counterexample to the Hamiltonian Seifert conjecture in dimension 5 (i. e. , for a compact hypersurface in $\mathbb{R}^6$). In 1997, V. Ginzburg improved the previous results so as to obtain a smooth proper function $H : \mathbb{R}^{2n} \to \mathbb{R}$, for $2n \geq 6$, with a regular level set on which the Hamiltonian flow has no closed orbits.