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Abstracts of Plenary and Invited Lectures

Section:

10. Partial Differential Equations

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Minimal regularity solutions of nonlinear wave equations

Recently there has been much interest in proving existence and uniqueness of solutions of nonlinear wave equations with low regularity initial data. One reason is that many equations from physics can be written as a system of nonlinear wave equations with a conserved energy norm. If one can prove local existence and uniqueness assuming only that the energy norm of initial data is bounded then global existence and uniqueness follow from the global energy bound. Therefore it is interesting to find the minimal amount of regularity of initial data needed to ensure local existence for the typical nonlinear wave equations.

In the lecture I will present counterexamples to local existence with low regularity data for the typical nonlinear wave equations. In the semi-linear case the counterexamples are sharp, in the sense that with slightly more regularity one can prove local existence. I will present in some detail new counterexamples for the quasi-linear case. It is natural to look for existence in Sobolev spaces, since the Sobolev norms are more or less the only norms that are preserved for a linear wave equation. The counterexamples involve constructing a solution that develops a singularity along a characteristic for all positive times. In the quasi-linear case it also involves controling the geometry of the characteristic set. The norm is initially bounded but becomes infinite for all positive times, contradicting the existence of a solution in the Sobolev space associated with the norm. The counterexamples are half a derivative more regular than what is predicted by a scaling argument. This argument use the fact that the equations are invariant under a scaling to obtain a sequence of solutions for which initial data is bounded in an appropriate Sobolev norm. The counterexamples were not widely expected since for several nonlinear wave equations, one does, obtain local existence down to the regularity predicted by scaling.

On the other hand, the classical local existence theorems for nonlinear wave equations are not sharp in the semi-linear case. These are proved using just the energy inequality and Sobolev's embedding theorem. Recently these results were improved using space-time estimates for Fourier integral operators, know as Strichartz' estimates, and generalizations of these. There are many recent results in this field by for example Klainerman-Machedon, Grillakis, Ponce-Sideris and Tataru. In particular, Klainerman-Machedon proved that for equations satisfying the 'null condition', one can go down to the regularity predicted by the scaling argument mentioned above. I will present joint work with Sogge where we prove local existence with minimal regularity for a much studied simple class of model semi-linear wave equations. I should mention that there is a somewhat paralell development for KdV and nonlinear Schroedinger equations, with work of for example Bourgain and Kenig-Ponce-Vega.

Whereas the techniques of harmonic analysis were essential in improving the local existence results, the Strichartz estimates are not the best possible global estimates since they don't catch the right decay as time tends to infinity if the initial data has compact support. The classical method, introduced by Klainerman, to prove global existence for small initial data is to use the energy method with the vector fields coming from the invariances of the equation. However, this method requires quite a lot of regularity of initial data and also the energy method alone doesn't give optimal estimates for the solution since it is an estimate for derivatives. I will present joint work with Georgiev and Sogge where we obtain better global estimates using techniques from harmonic analysis taking into account the invariances or symmetries of the wave equation. We obtaine estimates with mixed norms in the angular and spherical variables, with Sogge, and weighted Strichartz' estimates with Georgiev and Sogge. Using these new estimates we prove that a certain class of semi-linear wave equations have global existence in all space dimensions. This was a conjecture by Strauss, following an initial result by John.