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10. Partial Differential Equations

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Blow-up Phenomenon For Critical Nonlinear Schrödinger and Zakharov Equations.

We are interested in time singularity formation in dispersive Hamiltonian systems with infinite speed of propagation such as the nonlinear Schrödinger equation (NLS). We will consider the critical nonlinear Schrödinger equation and critical Zakharov equation (NLS coupled with a wave equation) in whole space. These equations appear in particular in plasma physics, laser beam propagation, etc. .. The Cauchy problem of these equations is locally well-posed in energy space and the question is to describe the singularity formation in this space.

For NLS, a virial identity yields existence of singular solutions. The notion of criticallity is related to a conformal invariance of the equation. We first see that the space L^2 (invariant by the flow) plays a crucial role. There is an L^2 concentration of the solution at the blow-up time of an amount at least greater than a universal number M_0. From the conformal invariance and the existence of a periodic solution Q, we have an explicit blow-up solution S such that $|S|_{L^2} = M_0$. Note that other blow-up solutions can be generated from S.
We first show that up to the invariance of the equation, S is the unique blow-up solution of minimal mass M_0. In particular, solutions at the critical mass level M_0 or below are, up to the invariance of the equation, either Q, S, or a solution defined for all time and behaving like a linear solution as the time goes to infinity.
We then consider the question of continuation of the solution after the blow-up time in the case of the minimal blow-up solution S. We show that the time singularity is in some sense unstable in time and there exists regular continuations of the solution after the blow-up time. However, the deterministic character(through one parameter in S^1) is lost through the blow-up and we describe how this occurs.

We are now interested in the structural stability of such qualitative properties of blow-up solutions, considering the Zakharov equations which can be seen as a more realistic physical approximation near the singularity. Note that there are no conformal invariance nor simple arguments for obstruction to global existence.

First, from a virial type identity, existence of a large class of blow-up solutions can be exhibited (in finite or infinite time). L^2 concentration properties of the blow-up solutions can be proved.

From a bifurcation argument at infinity (related to the solution S for critical NLS), we prove the existence of explicit blow-up solutions for Zakharov equations with the same qualitative structure as S for NLS equation. Existence of such solutions will have various conscequences (for example, instability of periodic solutions).

To conclude, the blow-up rate for any blow-up solution of the Zakharov equation (even when the approximation parameter related to NLS is small) is greater or equal to the one given by S. In particular, the blow-up rate numerically observed for blow-up solutions of NLS (different from the one given by S) is unstable under perturbations related to the Zakharov equation.