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On the problem of stability for near to integrable Hamiltonian systems

The problems addressed are concerned with the “stability” of near to integrable Hamiltonian systems in the light of the strong development of the quantitative methods of perturbation theory in the second half of this century. The word “stability” is used here in a quite wide sense, which includes a considerable weakening of the traditional concept of stability, as widely studied, e. g., by Lyapounov. The basis of the results that I'm going to illustrate is on the one hand the celebrated KAM theory on persistence of conditionally periodic motions, and, on the other hand, the Nekhoroshev's theorem on stability over exponential times.

Let me first illustrate the framework. I will consider the problem that Poincaré has called “the general problem of dynamics”, namely a canonical system of differential equations with Hamiltonian $H(p,q) = h(p) + \epsilon f(p,q,\epsilon)$; here $p \in \mathbf{R}^n$ and $q \in \mathbf{T}^n$ are action–angle variables, n is the number of degrees of freedom and ϵ is a small parameter. It is well known that for $\epsilon = 0$ the orbits are dense on invariant tori $\mathbf{T}^k$, with some positive $k \leq n$, which are subsets of a torus $p = \text{const}$. The motion is either quasiperiodic or periodic according to the existence of resonance relations among the unperturbed frequencies $\omega(p) = \frac{\partial h}{\partial q}$.

The main achievement of KAM theorem is the proof that for most initial data, characterized by strongly non–resonant frequencies, the motion is still quasi periodic, the orbit lying on an invariant torus $\mathbf{T}^n$ which is close to the unperturbed one. This relaxes the condition for stability that requires the result to hold for an open set of initial data: the complement of the surviving invariant tori is open and dense, but has small measure.

The theory of Nekhoroshev aims at proving that the action variables satisfy a bound $|p(t) - p(0)| < B\epsilon^b$ for all times $|t| < A\exp(1/\epsilon^a)$, with positive constants A, C, $a < 1$ and $b < 1$. This relaxes the condition that the stability result should hold for an infinite time interval; however, the exponential dependence of the estimated stability time on the inverse of the perturbation makes the result interesting for physical systems. Following Littlewood, I will call this phenomenon *exponential stability.*

My purpose here is to report on some progress made on this subject during the last decade. Most of the attention will be concentrated on the exponential stability. I will address in particular the following points: (a) the actual applicability of the concept of exponential stability to physical systems; (b) the extension of the concept of exponential stability to system with a very large number of degrees of freedom, and possibly to infinite systems; (c) the relations between the existence of KAM invariant tori and the exponential stability of Nekhoroshev's theory.

The relevance of the exponential stability for physical systems depends on the size of the perturbations that is actually allowed. However, the analytical estimates that are available are usually very pessimistic, unable to give realistic results. For a practical application it is more convenient to perform explicitly the perturbation expansions up to some finite order in the perturbation parameter, and to exploit the asymptotic character of the resulting series. This requires a lot of algebraic manipulation that the available computers do allow, so that the method may be effectively applied at least to simple cases. In this spirit several attempts have been made to investigate the stability of the Lagrangian triangular equilibria of the restricted problem of three bodies in the Sun–Jupiter case. The best result currently available is that there is a rather big neighbourhood of the equilibrium which is stable for a time interval as large as the estimated age of the universe; this neighbourhood is big enough to include a few of the known asteroids.

The actual applicability of the exponential estimates to systems with a large number of degrees of freedom is questionable because the available values of the constants entering the expression above depend in a bad manner on the number of degrees of freedom. The worse dependence is that of the constant a in $\exp(1/\epsilon^a)$: one has typically $a = O(1/n)$. The latter estimate has been shown to be optimal, in general. However, a further relaxation of the concept of stability shows that some version of the theory may still be applied. Precisely, instead of looking for an almost constant value of *all* the actions of the system one attempts to prove only the existence of at least one almost invariant quantity. A significant case concerns systems that satisfy the following requirements: (i) there a natural splitting into two distinct subsystems characterized by well separated frequencies or characteristic time intervals, and (ii) the two subsystems are weakly coupled. A first example is a FPU type chain with alternating heavy and light mass points: the spectrum splits naturally into two different parts, called the acoustic and the optical part, with a quite large gap between the frequencies. In this case the perturbation parameter is the ratio ω_-/ω_+ between the maximal acoustic frequency and the minimal optical frequency. A second example is, for instance, a system of identical diatomic molecules moving on a line and interacting with a short range analytic potential. The perturbation parameter is the ratio τ/ω between the typical interaction time during a collision and the internal frequency of vibration of the molecules.

In both these examples the high frequencies modes turn out to be (at least approximately) equal, so that the high frequency subsystem is completely resonant. Precisely the complete resonance allows us to prove that the exchange of energy between the two subsystems is practically frozen for a time interval $\exp(1/\epsilon)$, i. e., that the exponent a in the exponential estimate is 1, not $1/n$. This fact may be very relevant for applications to Statistical Mechanics.

The relations between the existence of KAM tori and the phenomena of exponential stability may be investigated by considering the neighbourhood of an invariant KAM torus. This leads to the concept of superexponential stability. Here is the argument, short but essentially complete. According to Kolmogorov's approach, the neighbourhood of the torus is described by an Hamiltonian which is very similar to the Hamiltonian describing the neighbourhood of an elliptic equilibrium, with nonresonant frequencies. The perturbation parameter is the distance ρ from the invariant torus. On the other hand, a local application of Nekhoroshev's theorem shows that the Hamiltonian may be given the form of the general problem of dynamics with a perturbation that is $O\big(\exp(-1/\rho^a)\big)$. Therefore, using the global formulation of Nekhoroshev's theorem one proves that the neighbourhood of a torus is exponentially stable in $O\big(\exp(-1/\rho^a)\big)$, that is, it is stable up to a time $O\big[\exp\big(\exp(1/\rho^a)\big)\big]$. The name "superexponential stability" refers to the double exponential appearing in the estimate.