ICM 1998, Berlin, Aug. 18–27

Abstracts of Plenary and Invited Lectures

Section:

12. Probability and Statistics

1991 MS Classification:

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Random and Deterministic Perturbations of Nonlinear Oscillators

We consider perturbations of an oscillator with one degree of freedom:

$$\ddot{q}_t^\epsilon + f(q_t^\epsilon) = \epsilon b(\dot{q}_t^\epsilon, q_t^\epsilon) + \sqrt{\epsilon}\sigma(\dot{q}_t^\epsilon, q_t^\epsilon) \circ \dot{W}_t, \quad q_0^\epsilon = q \in R^1.$$

Here W_t is the Wiener process in R^1, $0 < \epsilon << 1$. A typical example of the deterministic part of the perturbation is a small friction: $b(q, \dot{q}) = -\beta\dot{q}$. Let $F(q) = \int_0^q f(s)ds$, $\dot{q} = p$, $H(p,q) = p^2/2 + F(q)$. We assume that the potential $F(q)$ is generic (this means that $F(q)$ has a finite number of non-degenerate critical points) and $\lim_{|q|\to\infty} F(q) = \infty$. Let Γ be the graph homeomorphic to the set of connected components of the level sets of the Hamiltonian $H(p,q)$ provided with the natural topology: $\Gamma = \{I_1, ..., I_n; O_1, ...O_m\}$. Each vertex O_i belongs either to one edge (exterior vertex) or to three edges (interior vertex). A point $y \in \Gamma$ can be characterized by two coordinates (H, k), where k is the number of the edge containing y, and H is the value of the Hamiltonian on the level-set component corresponding to y.

Define a mapping $Y : \mathrm{R}^2 \to \Gamma$ so that $Y(x) = (H(x), k(x))$, $x \in R^2$, where $k(x)$ is the number of the edge containing the point corresponding to the trajectory of the non-perturbed system ($\epsilon = 0$) starting at $x \in R^2$. The function $k(x)$, as well as $H(x)$, is a first integral for the non-perturbed system. The pair $X_t^\epsilon = (p_t^\epsilon, \quad q_t^\epsilon), p_t^\epsilon = \dot{q}_t^\epsilon$, forms a diffusion process. The long-time behavior of this process as $\epsilon \downarrow 0$ is determined by the averaging principle. The slow component of X_t^ϵ is the projection $Y(X_t^\epsilon)$ of X_t^ϵ on Γ. We prove that the processes $Y_t^\epsilon = Y(X_{t/\epsilon}^\epsilon)$ converge weakly in the space of continuous functions on Γ to a diffusion process Y_t on the graph $\Gamma = \{I_1, ..., I_n; O_1, ...O_m\}$.

Diffusion processes on a graph Γ are defined by a family of second order differential operators $L_1, ..L_n$ and by gluing conditions at the vertices. The operator L_k defines Y_t inside $I_k \subset \Gamma$. The gluing conditions at a vertex $O_i \in \Gamma$ define, roughly speaking, what the trajectory is doing after is touches O_i. To describe the limiting process Y_t, denote by $G_k(z)$ the domain in R^2 bounded by $C_k(z) = Y^{-1}(z,k)$ if (z,k) is not a vertex. Let $S_k(z)$ be the area of $G_k(z)$. One can consider $S_k(z)$ as a function on Γ. This function has, in general, discontinuities at the interior vertices. If O_i is an interior vertex and $I_{i_1}, I_{i_2}, I_{i_3}$ are the edges attached to O_i, denote $S_j(O_i)$ the limit of $S_{i_j}(z)$ as the point $(z, i_j) \in I_{i_j}$ approaches O_i. Then in the case $b(p,q) = 0$, $\sigma =$ unit matrix, the operators L_k defining the process on I_k have the form

$$L_k = \frac{1}{2S_k'(z)} \frac{d}{dz}(S_k(z)\frac{d}{dz}).$$

A smooth inside the edges and continuous on Γ function $u(y), y \in \Gamma$, belongs to the domain of the generator A of the limiting process Y_t, if and only if $Au(y)$ is continuous on Γ and at any interior vertex $O_i \in \Gamma$ the following gluing condition is satisfied

$$\sum_{j=1}^{3} S_j(O_i) D_j u(O_i) = 0.$$

Here D_j is the operator of differentiation in the coordinate H along I_{i_j} attached to $O_i, j = 1, 2, 3$. The operators L_k and the gluing conditions at the vertices define the limiting process in a unique way. The characteristics of the limiting process on Γ are calculated also for any $\sigma(p,q)$ and $b(p,q)$. Let now $\sigma \equiv 0$, so that just deterministic perturbations are considered. it turns out that the slow component $Y(X^{\epsilon}_{t/\epsilon})$ of the deterministically perturbed system in any reasonable sense converges as $\epsilon \downarrow 0$ to a stochastic process on Γ. In this case the limiting process $Y_t = (z_t, k_t)$ is deterministic inside each edge I_k:

$$\begin{aligned} \dot{Y}_t =& b_k(Y_t), Y_t \in I_k, \\ b_k(z) =& \frac{1}{S_k'(z)} \int_{G_k(z)} \frac{\partial b(p,q)}{\partial p} dpdq. \end{aligned}$$

At an interior vertex O_i the trajectory branches with certain probabilities between the edges attached to the vertex. The branching probabilities are expressed through the function $b(p,q)$ and the curve $Y^{-1}(O_i)$. To give rigorous sense to this statement, we consider random perturbations of the deterministic system $\dot{X}^{\epsilon}_{t/\epsilon} = (p^{\epsilon}_{t/\epsilon}, q^{\epsilon}_{t/\epsilon})$ and then calculate the limit when first $\epsilon \downarrow 0$ and then the random perturbations tend to zero. It is important to underline that the limit is the same for a wide class of random perturbations of the equation or of the initial conditions. So that the structure of the limiting stochastic process is an intrinsic property of the oscillator and its deterministic perturbations, but not of the random perturbations which we use for regularization of the problem. We also will discuss some similar problems for systems with many degrees of freedom.