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Section:

12. Probability and Statistics

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Within and beyond the reach of Brownian innovation

A discrete-time random process is a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables; the corresponding distribution is a probability measure μ on the space $\mathbb{R}^\mathbb{N}$ of all sequences $(x_1, x_2, \dots)$ of real numbers. In particular, independent random variables U_n uniform on $(0,1)$ correspond to the product-measure $\mathrm{mes}^\mathbb{N} = \mathrm{mes} \otimes \mathrm{mes} \otimes \dots$ where mes is Lebesgue measure on $(0,1)$. An arbitrary μ is the image of $\mathrm{mes}^\mathbb{N}$ under some (non-unique) transformation $f : (0,1)^\mathbb{N} \to \mathbb{R}^\mathbb{N}$ of the form

$$\begin{aligned} &f(u_1, u_2, \dots) = (x_1, x_2, \dots) \quad \text{when} \\ &x_1 = f_1(u_1),\ x_2 = f_2(u_2, x_1),\ x_3 = f_3(u_3, x_2, x_1),\ \dots \end{aligned} \tag{1}$$

Thus, arbitrary random variables X_n can be produced from "standard" random variables U_n by taking $X_n = f_n(U_n, X_{n-1}, \dots, X_1)$, which is an elementary case of *innovation.* A non-elementary case of innovation is a stochastic differential equation (SDE)

$$dX(t) = \sigma\Big(t,\ \big(X(s)\big)_{s\in[0,t]}\Big)\, dB(t) + v\Big(t,\ \big(X(s)\big)_{s\in[0,t]}\Big)\, dt\,. \tag{2}$$

Here the idea is to produce a continuous-time random process $X(\cdot)$ from the Brownian motion $B(\cdot)$, the latter being basically a vast collection of small independent increments $B(t+\Delta t) - B(t)$. The common feature of the two forms of innovation is as follows. Equation (1) implies

$$\mathbb{E}\big(\, \phi(X_1, X_2, \dots) \,\big|\, U_1, \dots, U_n \,\big) = \mathbb{E}\big(\, \phi(X_1, X_2, \dots) \,\big|\, X_1, \dots X_n \,\big) \tag{3}$$

for all $n \in \mathbb{N}$ and all bounded Borel functions $\phi : \mathbb{R}^\mathbb{N} \to \mathbb{R}$, while for (2),

$$\mathbb{E}\Big(\phi\big(\big(X(s)\big)_{s\in[0,\infty)} \big) \Big| \big(B(s)\big)_{s\in[0,t]} \Big) = \mathbb{E}\Big(\phi\big(\big(X(s)\big)_{s\in[0,\infty)} \big) \Big| \big(X(s)\big)_{s\in[0,t]} \Big) \tag{4}$$

for all $t \in (0,\infty)$ and all bounded Borel functions ϕ on the relevant space of functions. In the modern jargon of *filtrations* (increasing families of σ-fields) we say that the filtration of (X_n) is *immersed* into the filtration of (U_n), and similarly, the filtration of $X(\cdot)$ is immersed into the filtration of $B(\cdot)$. Unexpected phenomena emerge, however, when passing from discrete time (1) to continuous time (2). Indeed, something strange already happens for discrete time if we extend the past back infinitely far; $\big(X_n\big)_{n\in-\mathbb{N}} = (\dots, X_{-2}, X_{-1})$. For example, consider the following very simple equation of the type (1) but for $n \in -\mathbb{N}$:

$$X_n = \lambda U_n + X_{n-1} \qquad (\mathrm{mod}\ 1); \tag{5}$$

that is, $X_n \in (0,1)$ and $X_n - (\lambda U_n + X_{n-1}) \in \mathbb{Z}$. (Here $\lambda \in (0,\infty)$ is a parameter, and once again, the U_n are independent and uniform on $(0,1)$.) Equation (5) determines uniquely the joint distribution of all X_n, but fails to determine a function $f : (0,1)^{-\mathbb{N}} \to \mathbb{R}^{-\mathbb{N}}$ such that $(X_n)_{n\in-\mathbb{N}} = f\big((U_n)_{n\in-\mathbb{N}}\big)$. In fact, (5) implies that X_{-1} is independent of $(U_n)_{n\in-\mathbb{N}}$, and the filtration of (X_n) is not a part of the filtration of (U_n). We call (X_n) a *weak solution* of (5). It is not an innovation! However, an innovation for (X_n) exists. For $\lambda = 1$ it is simple: just take $X_n = U_n$ instead of (5).
There exists $(X_n)_{n\in-\mathbb{N}}$ admitting no innovation (Vershik 1970); its filtration is *non-standard.* The phenomenon of (5) has its continuous-time counterpart, an SDE that fails to innovate (Tsirelson 1975, investigated further by Beneš, Stroock, Yor, Le Gall), though its (weak) solution admits another innovation (Émery and Schachermayer 1998). There is an SDE whose solution admits no innovation (Dubins, Feldman, Smorodinsky, Tsirelson 1994); its filtration is *non-Brownian.* Brownian motions in graphs (treated as one-dimensional topological spaces), known as Walsh's Brownian motions or spider-martingales, generate non-Brownian filtrations (Tsirelson 1996), differing from Brownian filtrations by the *splitting multiplicity* (Barlow, Émery, Knight, Song, Yor 1997). There is a pure martingale generating a non-Brownian filtration (Émery and Schachermayer 1998). The SDE for the sticky Brownian motion generates a non-Brownian filtration (Tsirelson and Warren 1998).
We know of three filtration invariants: dimension (understood long ago, integer-valued or ∞), splitting multiplicity (integer-valued or ∞), and *cosiness* (two-valued). The n-dimensional Brownian motion ($n = 1, 2, \dots; \infty$) generates a cosy filtration of dimension n and splitting multiplicity 2. The latter, combined with well-known facts about submartingales, subharmonic functions and Green functions, proves (Tsirelson 1997) a conjecture of Bishop (1991): in any dimension, the natural projection of the Martin boundary to the topological boundary is at most 2 to 1 almost everywhere. Splitting multiplicity is robust under measure changes and invertible time changes, while cosiness is not. Splitting multiplicity > 2 implies non-cosiness. All known non-Brownian filtrations of instant dimension 1 are non-cosy.
A Brownian filtration, by definition, is generated by a finite or countable collection of independent Brownian motions. On the other hand, a Brownian motion in a topological group G is defined as a continuous G-valued random process with stationary independent increments on the left, starting from the unit of G. Thus, a filtration is Brownian if and only if it is generated by *some* Brownian motion in the Hilbert space (separable, infinite-dimensional). Every Brownian motion in a Lie group generates a Brown-

ian filtration (Yosida 1952). The same holds for some infinite-dimensional groups: the group of diffeomorphisms (Baxendale 1984) and the group of unitary operators (Tsirelson 1998). The proof of the latter involves continuous tensor products (Feldman 1971, Arveson 1989, Tsirelson and Vershik 1995) and continuous quantum measurements (Davies 1976).

Problem 1. Is there a Brownian motion in *some* Polish group, that generates a non-Brownian filtration?

Here, "non-Brownian" may be replaced with "non-cosy", since a cosy filtration generated by a Brownian motion in a group is necessarily Brownian (Tsirelson 1998). If the answer is positive, we face another problem:

Problem 2. Does every random process admit an innovation by some Brownian motion in some Polish group?

References:

M. Yor, *Some aspects of Brownian motion, part II,* Birkhauser 1997. (See Chapter 17.)

B. Tsirelson, Triple points: from non-Brownian filtrations to harmonic measures. *Geom. Funct. Anal. (GAFA)* **7** (1997), 1096–1142.