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Section:

13. Combinatorics

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Ordering the symmetric group: Why use Iwahori-Hecke algebras?

As a Coxeter group, the symmetric group is generated by simple transpositions $\sigma_i := (i, i+1)$. Taking subwords of *reduced* products of σ_i induces an order on permutations, the *Bruhat order*.
Instead of enumerating all subwords, one can replace decompositions $\sigma_i \sigma_j \cdots \sigma_h$ by products $(1+\pi_i)(1+\pi_j)\cdots(1+\pi_h)$ and expand them in the *0-Hecke algebra* (whose generators π_i satisfy the braid relations, and $\pi_i^2 = -\pi_i$). One then obtains the sum of all permutations below a given one.
Another method consists in taking left factors of permutations (as words in $\{1, 2, \dots\}$), sorting them and comparing them componentwise. This amounts to associating to a permutation, a Young tableau encoding all the cosets μP, where $\mu \in S_n$ and where the P's are all the maximal parabolic subgroups. One can instead use double cosets $P\,\mu\,P'$ and associate to a permutation the minimum representatives β of all the double cosets, such that β is below μ in the Bruhat order: $\mathcal{S}_n \ni \mu \mapsto B(\mu) := \{\beta : \beta \le \mu\}$. The order becomes inclusion of sets :

$$\mu \le \zeta \Leftrightarrow B(\mu) \subseteq B(\zeta) \ .$$

This furnishes an embedding of the symmetric group, as an ordered set, into the (distributive) lattice of *alternating sign matrices*.
However, being able to compare permutations does not provide a way to describe intervals for the Bruhat order. For example, since the symmetric group is a rank-poset, combinatorists would like to describe the *Poincaré polynomial* of an initial interval $[1, \mu]$ (for geometers this polynomial is equal to the number of points of a Schubert variety over the finite field F_q). For some permutations (which correspond to non-singular Schubert varieties) this polynomial is known to be symmetric and to factor into q-integers $1 + q + \cdots + q^k$. One can lift such a factorization, not to the group algebra of $\mathcal{S}_n$, but to the *Iwahori-Hecke algebra* (the generators T_i satisfy the braid relations, and $(T_i + 1)(T_i - q) = 0$). For example the element

$$c_\mu := \sum\nolimits_{\mu \le [4312]} (-q)^{\ell(\mu)} T_\mu = (T_3 - q)(T_2 - \frac{q^2}{1+q})(T_1 - \frac{q^2}{1+q})(T_3 - q)(T_2 - q)$$

becomes, under the specialization $T_1 = T_2 = T_3 = -1$, equal to $-(1+q)(1+q+q^2)^2$.

This special element c_μ belongs in fact to the *Kazhdan-Lusztig basis* which is a distinguished linear basis of the Hecke algebra of fundamental importance in representation theory and in geometry. The coefficients of the matrix expressing the Kazhdan-Lusztig basis into the standard basis T_μ are called *Kazhdan-Lusztig polynomials.* These are polynomials in q with (small) integral positive coefficients, that are, unfortunately, hard to obtain, despite their elementary recursions. It is still a challenge to provide a combinatorial description of these polynomials. Such a description would have applications to properties of the Bruhat order, to desingularizations of Schubert varieties, multiplicities of Verma modules, &c.

In the case of the minimum representatives β which code the order (and correspond to *determinantal varieties*, i. e. varieties defined by the vanishing of minors of a given order of a generic matrix) the Kazhdan-Lusztig polynomials are just Gauss polynomials, also called *q-binomial coefficients.*

We shall show that the above stated embedding of the symmetric group into a lattice furnishes an explicit combinatorial description of Kazhdan-Lusztig polynomials in the case of "3412-avoiding" permutations.

References:

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