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## Abstracts of Plenary and Invited Lectures

### Section:

## 13. Combinatorics

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**Finite Geometries, Varieties and Codes**

In recent years there has been an increasing interest in finite projective spaces, and important applications to practical topics such as coding theory and design of experiments have made the field even more attractive. In 1935, Yates, in a lecture to the Royal Statistical Society, introduced the notion of design, thus showing how finite geometries can be applied to the planning of experiments. A few years later the statistician Bose started with his investigations on graph theory, design theory and finite projective spaces, using mainly pure combinatorial arguments. Another great pioneer in finite projective geometry was the Italian geometer Beniamino Segre. His celebrated result of 1954 stating that in the projective plane $\mathrm{PG}(2,q)$ over the Galois field $\mathrm{GF}(q)$ with $q$ odd, every set of $q+1$ points, no three of which are collinear, is a conic stimulated the enthusiasm of many young geometers. Finally, the fundamental and deep work in the last four decades on polar spaces, generalized polygons, and, more generally, incidence geometry, in the first place by Tits, but also by Buekenhout, Kantor, Shult, Veldkamp and others, gave a new dimension to finite geometry. It is my intention to mention a few important and elegant theorems, to say something about the used techniques and the relation with other fields, and to mention some conjectures.

First I will speak on the geometry of $\mathrm{PG}(2,q)$, that is, the projective plane over the finite field $\mathrm{GF}(q)$. To begin with it is my purpose to show how classical algebraic curves in $\mathrm{PG}(2,q)$ can be characterized in pure combinatorial terms. I will illustrate this with one theorem on conics and one on Hermitian curves. Next, I will mention a characterization of Hermitian curves in terms of algebraic geometry, that is, from the start we assume that our pointset in $\mathrm{PG}(2,q)$ is an algebraic curve. In the proofs of these three theorems classical projective geometry, finite algebraic geometry, finite field theory and counting arguments were used. Relying on completely different techniques Ball, Blokhuis and Mazzocca proved in 1997 the longstanding conjecture saying that for $q$ odd pointsets $P$ of $\mathrm{PG}(2,q)$ with the property that each line of $\mathrm{PG}(2,q)$ either contains 0 or $m$ points of $P$, $m$ a constant with $1 < m < q$, do not exist.

Next I will speak about the geometry of $\mathrm{PG}(n,q)$. If $\mathcal{V}$ is a "classical" algebraic variety in $\mathrm{PG}(n,q)$, e. g. a quadric, a Hermitian variety, a Veronese variety, then a first approach is to characterize $\mathcal{V}$ either as a subset of

PG$(n,q)$ which intersects certain subspaces of PG$(n,q)$ in sets with cardinalities in some range or as a subset of PG$(n,q)$ whose points satisfy certain linear independence conditions. One characterization of the first type will be given, while in the section on arcs and codes we will show that (subsets) of normal rational curves can be characterized by one simple independence condition on the points. A second approach is that where the object is described as an incidence structure satisfying certain properties; here the object is not a priori embedded in a projective space, even the finite field is in many cases a priori absent. This will be illustrated with some beautiful theorems on inversive (or Möbius) planes, polar spaces and Shult spaces.

Finally there will be a section on $k$-arcs in PG$(n,q)$ and on linear Maximum Distance Separable codes (MDS codes), where the interplay between finite projective geometry, coding theory and algebraic geometry is particularly present. A $k$-arc $K$ in PG$(n,q), n \geq 2$, is a set of $k$ points of PG$(n,q)$ with $k \geq n+1$ such that no $n+1$ points of $K$ lie in a hyperplane of PG$(n,q)$. Many deep theorems on $k$-arcs were and are still obtained using finite projective and algebraic geometry. A linear MDS code is a linear code for which we have equality in the Singleton bound; in their famous book on coding theory MacWilliams and Sloane [1977] introduce their chapter on MDS codes as "one of the most fascinating in all of coding theory". Now it appears that MDS codes of dimension at least three and $k$-arcs are equivalent objects, so that all results on $k$-arcs can be immediately translated in terms of MDS codes.

In an appendix, as an example of brand new research in the field, I will mention a problem on representing finite projective planes into the lineset of PG$(n,q)$, its complete solution, and an application to generalized hexagons.

**References**

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